THE DISCRETE UNIVERSALITY OF THE PERIODIC HURWITZ ZETA-FUNCTION

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The paper contains a survey on continuous and discrete universality theorems for periodic zeta-functions. A sketch of the proof in the case of the discrete universality for the periodic Hurwitz zeta-function is given. Also, joint generalizations for periodic Hurwitz zeta-functions are formulated.

Introduction

Denote by $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all non-negative integers, positive integers, integers, real and complex numbers, respectively. Let $\mathfrak{A} = \{a_m : m \in \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, $0 < \alpha \leqslant 1$. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{A})$, $s = \sigma + it$, is defined, for $\sigma > 1$, by the series

$$\zeta(s, \alpha; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{a_m}{(m + \alpha)^s}. \tag{1}$$

It follows from the periodicity of the sequence $\mathfrak{A}$ that, for $\sigma > 1$,

$$\zeta(s, \alpha; \mathfrak{A}) = \frac{1}{k^s} \sum_{l=0}^{k-1} a_l \zeta\Big(s, \frac{l + \alpha}{k}\Big), \tag{2}$$

where, for $\beta \in \mathbb{R}$, $0 < \beta \leqslant 1$, $\zeta(s, \beta)$ is the classical Hurwitz zeta-function. We recall that in the half-plane $\sigma > 1$ the Hurwitz zeta-function is defined by

$$\zeta(s, \beta) = \sum_{m=0}^{\infty} \frac{1}{(m + \beta)^s}.$$

Moreover, the Hurwitz zeta-function is analytically continuable to the whole complex plane, except for a simple pole at $s = 1$ with residue 1. Therefore, in view of (2) we have that the periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{A})$ is also analytically continuable to the whole s-plane, except for a simple pole at $s = 1$ with residue

$$a \stackrel{\text{def}}{=} \frac{1}{k} \sum_{l=0}^{k-1} a_l.$$

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If $a = 0$, then $\zeta(s, \alpha; \mathfrak{A})$ is an entire function.

By (1), if $a_m = 1$ for all $m \in \mathbb{N}_0$, then $\zeta(s, \alpha; \mathfrak{A}) = \zeta(s, \alpha)$. Thus, the function $\zeta(s, \alpha; \mathfrak{A})$ is a generalization of the classical Hurwitz zeta-function.

Let $\lambda \in \mathbb{R}$. Then the Lerch zeta-function $L(\lambda, \alpha, s)$, for $\sigma > 1$, is given by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s},$$

and by analytic continuation elsewhere. Clearly, $L(\frac{l}{k}, \alpha, s)$ is a particular case of the function $\zeta(s, \alpha; \mathfrak{A})$.

1. Universality

A property of one mathematical object to have the influence for a large class of other mathematical objects is understand as the universality. In analysis, the first universal object was found by M. Fekete. He proved that there exists a real power series

$$\sum_{m=1}^{\infty} a_m x^m, \quad x \in [-1, 1],$$

such that, for every continuous function $g(x)$ on $[-1, 1]$, $g(0) = 0$, there exists a sequence of positive integers n_k, $\lim_{k \to \infty} n_k = +\infty$, such that

$$\lim_{k \to \infty} \sum_{m=1}^{n_k} a_m x^m = g(x)$$

uniformly in $x \in [-1, 1]$. Later, a numerous number of other universal in some sense objects were found, however, these objects were not explicitly given. As in the mentioned Fekete's theorem, only the existence of universal objects was proved. Only in 1975 S.M. Voronin obtained the universality of the Riemann zeta-function $\zeta(s)$, so this function is the first explicitly given universal object. We recall that

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

and $\zeta(s)$ has analytic continuation to the whole complex plane, except for a simple pole at $s = 1$ with residue 1. S.M. Voronin proved [11] that every analytic function can be approximated by shifts $\zeta(s + i\tau)$. More precisely, he obtained the following remarkable statement.

Theorem 1. *Let $0 < r < \frac{1}{4}$. Suppose that the function $f(s)$ is continuous and non-vanishing on the disc $\{s \in \mathbb{C} : |s| \leqslant \frac{1}{4}\}$, and analytic in the interior of this disc. Then, for every $\epsilon > 0$, there exists a real number $\tau = \tau(\epsilon)$ such that*

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \epsilon.$$

A big number of number theorists were interested in this remarkable Voronin's result. A. Reich, S.M. Gonek, B. Bagchi, A. Laurinčikas, K. Matsumoto, J. Steuding, W. Schwarz, H. Mishou, H. Bauer, H. Nagoshi, the author and others generalized the Voronin theorem for other zeta- and L-functions. At the moment, it is known that the majority of classical zeta and L-functions are universal in the Voronin sense. For example, Dirichlet L-functions, Dedekind zeta-functions, L-functions of elliptic curves over the field of rational numbers, zeta-functions of normalized eigenforms, some classes of Dirichlet series with multiplicative coefficients, and even some classes of general Dirichlet series

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s}, \quad a_m \in \mathbb{C}, \quad \lim_{m\to\infty} \lambda_m = +\infty,$$

have the universality property. By the Linnik–Ibragimov conjecture, all functions in some half-plane defined by Dirichlet series, analytically continuable to the left of the absolute convergence abscissa and satisfying some natural growth conditions, are universal in the Voronin sense.

Theorem 1 has a more general form. Denote by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$. Then Chapter 6 of [5] contains the following version of the Voronin theorem.

Theorem 2. *Suppose that K is a compact subset of the strip D with connected complement, and $f(s)$ is a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

Theorem 2 shows that the set of shifts $\zeta(s+i\tau)$ which approximate a given analytic function is sufficiently rich, its lower density is positive. On the other hand, Theorems 1 and 2 are non-effective in the sense that we do not know any value of τ such that

$$\sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon.$$

Theorem 2 follows in the following way. First, a limit theorem in the sense of weak convergence of probability measures in the space $H(D)$ of analytic on D functions is proved. This means that the probability measure

$$\frac{1}{T} \text{meas}\{\tau \in [0,T] : \zeta(s+i\tau) \in A\}, \quad A \in \mathcal{B}(H(D)), \tag{3}$$

where $\mathcal{B}(H(D))$ denotes the class of Borel sets of the space $H(D)$, converges weakly to some probability measure P on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$. After this, it is proved that the support of P is the set

$$\{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Now this together with the Mergelyan theorem, see, for example, [12], on approximation of analytic functions by polynomials imply the theorem.

Theorem 2 has a continuous character, in it the shifts $\zeta(s+i\tau)$, where τ various continuously in the interval $[0,T]$, are investigated. It is also possible to consider the shifts $\zeta(s+imh)$, where $h>0$ is a fixed number and $m\in\mathbb{N}_0$. In this case, we have the discrete version of universality. We state a discrete analogue of Theorem 2.

Theorem 3. *Suppose that K and f(s) are the same as in the statement of Theorem 2. Then, for every* $\epsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\left\{0\leqslant l\leqslant N:\sup_{s\in K}|\zeta(s+ilh)-f(s)|<\varepsilon\right\}>0.$$

Theorem 3 is a modification of the results obtained in [9].

2. Continuous universality of $\zeta(s,\alpha;\mathfrak{A})$

The properties of the function $\zeta(s,\alpha;\mathfrak{A})$ are closely related to arithmetical nature of the parameter α. The proof of the limit theorem for the measure (3) is based on the fact that the system $\{\log p: p \text{ is prime}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$. In the case of the function $\zeta(s,\alpha;\mathfrak{A})$, we have a similar situation if α is transcendental. Then the system $\{\log(m+\alpha): m\in\mathbb{N}_0\}$ is linearly independent over $\mathbb{Q}$. In this case, the following theorem is true.

Theorem 4. *Suppose that* α *is transcendental. Then the probability measure*

$$\frac{1}{T}\text{meas}\{\tau\in[0,T]:\zeta(s+i\tau,\alpha;\mathfrak{A})\in A\},\quad A\in\mathcal{B}(H(D)),$$

converges weakly to some probability measure P_ζ *on* $(H(D),\mathcal{B}(H(D)))$ *as* $T\to\infty$.

The measure P_ζ is the distribution of one $H(D)$-valued random element related to the function $\zeta(s,\alpha;\mathfrak{A})$.

Let

$$\Omega=\prod_{m=0}^{\infty}\gamma_m,$$

where $\gamma_m=\{s\in\mathbb{C}:|s|=1\}$ for all $m\in\mathbb{N}_0$. The torus Ω is a compact topological Abelian group, therefore on $(\Omega,\mathcal{B}(\Omega))$ the probability Haar measure exists, and this gives the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. Denote by $\omega(m)$ the projection of $\omega\in\Omega$ to the coordinate space γ_m, $m\in\mathbb{N}$, and on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$ define the $H(D)$-valued random element $\zeta(s,\alpha,\omega;\mathfrak{A})$ by

$$\zeta(s,\alpha,\omega;\mathfrak{A})=\sum_{m=0}^{\infty}\frac{a_m\omega(m)}{(m+\alpha)^s}.$$

Then it turns out that the measure P_ζ coincides with the distribution of the random element $\zeta(s,\alpha,\omega;\mathfrak{A})$, i.e.

$$P_\zeta(A)=m_H(\omega\in\Omega:\zeta(s,\alpha,\omega;\mathfrak{A})\in A),\quad A\in\mathcal{B}(H(D)).$$

The proof of Theorem 4 is given in [1]. It is similar to that of a limit theorem for the Lerch zeta-function, see [4].

In [1], the case

$$\min_{0\leqslant m\leqslant k-1} |a_m| > 0 \tag{4}$$

was considered. The later condition was used to prove that the support of the measure P_ζ is the whole of $H(D)$. In [2], the positive density method developed in [7] was applied, and the requirement (4) was removed. This allowed to obtain the universality of the function $\zeta(s,\alpha;\mathfrak{A})$ for all periodic sequences $\mathfrak{A}$.

Theorem 5. *Suppose that* α *is transcendental. Let* K *be a compact subset of the strip* D *with connected complement, and let* $f(s)$ *be a continuous on* K *function which is analytic in the interior of* K*. Then, for every* $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha;\mathfrak{A}) - f(s)| < \epsilon\right\} > 0.$$

Note that in Theorem 5, differently from Theorem 2, the approximated function $f(s)$ is not necessarily non-vanishing. This is conditioned by non-existence of the Euler product for the function $\zeta(s,\alpha;\mathfrak{A})$.

3. Discrete universality of $\zeta(s,\alpha;\mathfrak{a})$

This section is devoted to a discrete version of Theorem 5. A theorem of such a kind was proved in [6].

Theorem 6. *Suppose that* α *is transcendental, and* $h > 0$ *is a fixed number such that* $\exp\{\frac{2\pi}{h}\}$ *is rational. Let* K *and* $f(s)$ *be the same as in the statement of Theorem 5. Then, for every* $\epsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant l \leqslant N : \sup_{s\in K} |\zeta(s+ilh,\alpha;\mathfrak{A}) - f(s)| < \epsilon\right\} > 0.$$

The transcendence of α and a condition for the number h are applied to obtain a probabilistic limit theorem in the space $H(D)$ for the function $\zeta(s,\alpha;\mathfrak{A})$. The proof of this theorem is based on a limit theorem on the torus Ω [6]. We give its proof there.

Lemma 7. *Suppose that* α *and* h *are the same as in the statement of Theorem 6. Then the probability measure*

$$Q_N(A) \stackrel{\text{def}}{=} \frac{1}{N+1}\#\left\{0 \leqslant l \leqslant N : \left((m+\alpha)^{-ilh} : m \in \mathbb{N}_0\right) \in A\right\}, \quad A \in \mathcal{B}(\Omega),$$

converges weakly to the Haar measure m_H *as* $N \to \infty$.

Proof. The dual group of the group Ω is

$$\mathcal{D} \stackrel{\text{def}}{=} \bigoplus_{m=0}^{\infty} \mathbb{Z}_m,$$

where $\mathbb{Z}_m = \mathbb{Z}$ for all $m \in \mathbb{N}_0$. An element $\underline{k} = \{k_m : m \in \mathbb{N}_0\} \in \mathcal{D}$, where only a finite number of integers k_m are distinct from zero, acts on Ω by

$$\omega \to \omega^{\underline{k}} = \prod_{m=0}^{\infty} \omega^{k_m}(m).$$

Therefore, the Fourier transform $g_N(\underline{k})$ of the measure Q_N is

$$\begin{aligned} g_N(\underline{k}) &= \int_{\Omega} \prod_{m=0}^{\infty} \omega^{k_m}(m) \mathrm{d}Q_N = \frac{1}{N+1} \sum_{l=0}^{N} \prod_{m=0}^{\infty} (m+\alpha)^{-ik_m lh} = \\ &= \frac{1}{N+1} \sum_{l=0}^{N} \exp\left\{-ilh \sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\}. \end{aligned} \tag{5}$$

The transcendence of α implies the irrationality of

$$\exp\left\{\sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\} = \prod_{m=0}^{\infty} (m+\alpha)^{k_m},$$

where only a finite numbers of integers $k_m \neq 0$. On the other hand, the number $\exp\{\frac{2\pi r}{h}\}$ is rational for all $r \in \mathbb{Z}$. Thus, for $\underline{k} \neq \underline{0}$,

$$\exp\left\{-ih \sum_{m=0}^{\infty} k_m \log(m+\alpha)\right\} \neq 1.$$

In view of this remark, (5) shows that

$$g_N(\underline{k}) = \begin{cases} 1, & \text{if } \underline{k} = \underline{0}, \\ \dfrac{1 - \exp\{-i(N+1)h \sum_{m=0}^{\infty} k_m \log(m+\alpha)\}}{(N+1)(1 - \exp\{-ih \sum_{m=0}^{\infty} k_m \log(m+\alpha)\})}, & \text{if } \underline{k} \neq \underline{0}. \end{cases}$$

Hence

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1, & \text{if } \underline{k} = \underline{0}, \\ 0, & \text{if } \underline{k} \neq \underline{0}, \end{cases} \tag{6}$$

and the lemma follows, since the Fourier transform of the Haar measure is the right-hand side of (6).

Now let

$$\zeta_n(s, \alpha; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{a_m \exp\{-(\frac{m+\alpha}{n+\alpha})^{\sigma_1}\}}{(m+\alpha)^s},$$

and, for $\omega \in \Omega$,

$$\zeta_n(s, \alpha, \omega; \mathfrak{A}) = \sum_{m=0}^{\infty} \frac{a_m \omega(m) \exp\{-(\frac{m+\alpha}{n+\alpha})^{\sigma_1}\}}{(m+\alpha)^s},$$

where $\sigma_1 > \frac{1}{2}$ is a fixed number. Then it is not difficult to see that the series for $\zeta_n(s, \alpha; \mathfrak{A})$ and $\zeta_n(s, \alpha, \omega; \mathfrak{A})$ both converge absolutely for $\sigma > \frac{1}{2}$. A simple application of Lemma 7 shows that the probability measures

$$P_{N,n}(A) = \frac{1}{N+1}\#\{0 \leqslant l \leqslant N : \zeta_n(s+ilh,\alpha;\mathfrak{A}) \in A\}, \quad A \in \mathcal{B}(H(D)), \tag{7}$$

and

$$\widehat{P}_{N,n}(A) = \frac{1}{N+1}\#\{0 \leqslant l \leqslant N : \zeta_n(s+ilh,\alpha,\omega;\mathfrak{A}) \in A\}, \quad A \in \mathcal{B}(H(D)), \tag{8}$$

both converge weakly to the same probability measure P_n on $(H(D),\mathcal{B}(H(D)))$ as $N \to \infty$.

Let K be a compact subset of the strip D. Then in [1] it was proved that, for transcendental α,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}|\zeta(s+i\tau,\alpha;\mathfrak{A}) - \zeta_n(s+i\tau,\alpha;\mathfrak{A})| = 0,$$

and, for almost all $\omega \in \Omega$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K}|\zeta(s+i\tau,\alpha,\omega;\mathfrak{A}) - \zeta_n(s+i\tau,\alpha,\omega;\mathfrak{A})| = 0,$$

The application of the Gallagher lemma, see [8], Lemma 1.4, leads to a discrete version of the above mean approximation. We find that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}|\zeta(s+ilh,\alpha;\mathfrak{A}) - \zeta_n(s+ilh,\alpha;\mathfrak{A})| = 0,$$

and, for almost all $\omega \in \Omega$,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}|\zeta(s+ilh,\alpha,\omega;\mathfrak{A}) - \zeta_n(s+ilh,\alpha,\omega;\mathfrak{A})| = 0.$$

Now the later two relations together with weak convergence of the probability measures $P_{N,n}$ and $\widehat{P}_{N,n}$ allow to prove that the probability measures

$$P_N(A) = \frac{1}{N+1}\#\{0 \leqslant l \leqslant N : \zeta(s+ilh,\alpha;\mathfrak{A}) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

and

$$\widehat{P}_N(A) = \frac{1}{N+1}\#\{0 \leqslant l \leqslant N : \zeta(s+ilh,\alpha,\omega;\mathfrak{A}) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

for almost all $\omega \in \Omega$, also converge weakly to the same probability measure P on $(H(D),\mathcal{B}(H(D)))$ as $N \to \infty$.

Define on Ω the measurable measure preserving transformation $\varphi_{h,\alpha}$ by

$$\varphi_{h,\alpha}(\omega) = \left((m+\alpha)^{-ih} : m \in \mathbb{N}_0\right)\omega.$$

Then the properties of the numbers α and h imply the ergodicity of $\varphi_{h,\alpha}$, and a simple application of the Birkhoff–Khintchine theorem, see [10], leads to a discrete limit theorem for the function $\zeta(s,\alpha;\mathfrak{A})$ [6].

Theorem 8. *Let* α *and* h *be the same as in the statement of Theorem 6. Then the probability measure*

$$\frac{1}{N+1}\#\{0 \leqslant l \leqslant N : \zeta(s+ilh, \alpha; \mathfrak{A}) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to P_ζ *as* $N \to \infty$.

Proof of Theorem 6. We already have seen in Section 3 that the support of the measure P_ζ is the whole $H(D)$. By the Mergelyan theorem there exists a polynomial $p(s)$ such that

$$\sup_{s\in K} |f(s) - p(s)| < \frac{\epsilon}{2}, \tag{9}$$

and $p(s)$ is an element of the support of P_ζ. Then, denoting

$$G = \left\{ g \in H(D) : \sup_{s\in K} |g(s) - p(s)| < \frac{\epsilon}{2} \right\},$$

we obtain by Theorem 8 and properties of the support that

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant l \leqslant N : \sup_{s\in K} |\zeta(s+ilh, \alpha; \mathfrak{A}) - p(s)| < \frac{\epsilon}{2}\right\} \geqslant P_\zeta(G) > 0.$$

This together with (9) prove the theorem.

Theorem 6 and Rouché's theorem yield a certain information on zeros of the function $\zeta(s, \alpha; \mathfrak{A})$.

Theorem 9. *Let* α *and* h *be the same as in the statement of Theorem 6. Then, for any* σ_1 *and* σ_2, $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, *and sufficiently large* N, *there exists a constant* $c = c(\sigma_1, \sigma_2, \alpha; \mathfrak{A}) > 0$ *such that the function* $\zeta(s+imh, \alpha; \mathfrak{A})$ *has a zero in the disc*

$$\left| s - \frac{\sigma_1 + \sigma_2}{2} \right| < \frac{\sigma_2 - \sigma_1}{2}$$

more than for cN *numbers* m, $0 \leqslant m \leqslant N$.

4. Joint case

We complete the paper with a joint generalization of Theorem 6. Let, for $j = 1, ..., r$, $\mathfrak{A}_j = \{a_{mj} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_j \in \mathbb{N}$, $0 < \alpha_j \leqslant 1$, and let $\zeta(s, \alpha_j; \mathfrak{A}_j)$ be the corresponding periodic Hurwitz zeta-function. Denote by k the least common multiple of the periods $k_1, ..., k_r$, and define

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{kr} \end{pmatrix}.$$

Then [3] contains the following joint universality theorem.

Theorem 10. *Suppose that* $\alpha_1, \dots, \alpha_r$ *are algebraically independent over* $\mathbb{Q}$, *and that* $\operatorname{rank}(A) = r$. *For each* $j = 1, \dots, r$, *let* K_j *be a compact subset of the strip* D *with connected complement, and let* $f_j(s)$ *be a continuous function on* K_j *which is analytic in the interior of* K_j. *Then, for every* $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + i\tau, \alpha_j; \mathfrak{A}_j) - f_j(s)| < \epsilon\right\} > 0.$$

Also, a discrete version of Theorem 10 can be proved in the following form.

Theorem 11. *Suppose that* $h > 0$ *is a fixed number such that* $\exp\{\frac{2\pi}{h}\}$ *is rational, and for* A, K_j *and* $f_j(s)$, $j = 1, \dots, r$, *the hypotheses of Theorem 10 are satisfied. Then, for every* $\epsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\left\{0 \leqslant l \leqslant N : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s + ilh, \alpha_j; \mathfrak{A}_j) - f_j(s)| < \epsilon\right\} > 0.$$

The proof of Theorem 11 will be published elsewhere.

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ДИСКРЕТНЫЕ УНИВЕРСАЛЬНОСТИ ДЛЯ ПЕРИОДИЧЕСКОЙ ДЗЕТА-ФУНКЦИИ ГУРВИЦА

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