VALUE DISTRIBUTION THEOREMS FOR THE ESTERMANN ZETA-FUNCTION

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In the paper, a survey of mean-value estimates, zero distribution, universality and limit theorems in the sense of weak convergence of probability measures for the Estermann zeta-function is presented.

Introduction

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Denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all positive integers, integers, real and complex numbers, respectively. For arbitrary $\alpha \in \mathbb{C}$ and $m \in \mathbb{N}$, the generalized divisor function $\sigma_\alpha(m)$ is defined by

$$\sigma_\alpha(m) = \sum_{d/m} d^\alpha.$$

We have that

$$\sigma_0(m) \ll_\varepsilon m^\varepsilon.$$

Since, $\sigma_\alpha(m) = m^\alpha \sigma_{-\alpha}(m)$, the estimate

$$\sigma_\alpha(m) \ll_\varepsilon m^{\varepsilon + \max(\mathrm{Re}\alpha, 0)} \tag{1}$$

is valid.

Let, as usual, $s = \sigma + it$ denote a complex variable, and $(k, l) = 1$. The Estermann zeta-function $E\left(s; \frac{k}{l}, \alpha\right)$, for $\sigma > \max(1 + \mathrm{Re}\alpha, 1)$, is defined by

$$E\left(s; \frac{k}{l}, \alpha\right) = \sum_{m=1}^\infty \frac{\sigma_\alpha(m)}{m^s} \exp\left\{2\pi i m \frac{k}{l}\right\}.$$

For analytic continuation of the function $E\left(s; \frac{k}{l}, \alpha\right)$ to the whole complex plane, we recall the definition of the Lerch zeta-function. Let $\lambda \in \mathbb{R}$ and $\beta \in \mathbb{R}, 0 < \beta \leqslant 1$. The Lerch zeta-function $L(\lambda, \beta, s)$, for $\sigma > 1$, is defined by

$$L(\lambda, \beta, s) = \sum_{m=0}^\infty \frac{e^{2\pi i m \lambda}}{(m+\beta)^s}.$$

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If $\lambda \notin \mathbb{Z}$, then $L(\lambda, \beta, s)$ is analytically continuable to an entire function, while for $\lambda \in \mathbb{Z}$, the function $L(\lambda, \beta, s)$ becomes the Hurwitz zeta-function

$$\zeta(s, \beta) = \sum_{m=0}^{\infty} \frac{1}{(m+\beta)^s}.$$

The function $\zeta(s, \beta)$ is meromophically continuable to the whole complex plane where it has a simple pole at $s = 1$ with residue 1.
It is not difficult to see that, for $\sigma > \max(\operatorname{Re}\alpha + 1, 1)$,

$$E\Big(s; \frac{k}{l}, \alpha\Big) = l^{\alpha - s} \sum_{\nu=1}^{l} \exp\Big\{2\pi i \frac{\nu k}{l}\Big\} L\Big(1, \frac{\nu}{l}, s - \alpha\Big) L\Big(\frac{\nu k}{l}, 1, s\Big). \tag{2}$$

The latter equation shows that the function $E\big(s; \frac{k}{l}, \alpha\big)$ is analytic in the whole complex plane, except for two simple poles at $s = 1$ and $s = 1 + \alpha$ if $\alpha \neq 0$, and a double pole $s = 1$ if $\alpha = 0$.
Let $\overline{k}$ be defined by $k\overline{k} \equiv 1 (\operatorname{mod} l)$. Then (2) and the functional equation for the Lerch zeta-function, see [11], imply the following functional equation for $E\big(s; \frac{k}{l}, \alpha\big)$.

$$E\Big(s; \frac{k}{l}, \alpha\Big) = \frac{1}{\pi}\Big(\frac{2\pi}{l}\Big)^{2s-1-\alpha} \Gamma(1-s)\Gamma(1+\alpha-s)\Big) \times$$
$$\Big(\cos\frac{\pi\alpha}{2} E\Big(1+\alpha-s; \frac{\overline{k}}{l}, \alpha\Big) - \cos\Big(\pi s - \frac{\pi\alpha}{2}\Big) E\Big(1+\alpha-s; \frac{\overline{k}}{l}, \alpha\Big)\Big).$$

Therefore, without loss of generality we may assume that $a \stackrel{\text{def}}{=} \operatorname{Re}\alpha \leqslant 0$.
Note that the function $E\big(s; \frac{k}{l}, \alpha\big)$, for $\alpha = 0$, was introduced by T. Estermann in [3]. The case of $\alpha \in [-1, 0]$ was considered in [9].
In the lecture, we discuss the following value distribution problems for the Estermann zeta-function:

- Mean square estimates
- Zero distribution
- Universality
- Probabilistic limit theorems

1. Mean square of $E\big(s; \frac{k}{l}, \alpha\big)$

Asymptotics and estimates for mean values of zeta-functions play an important role in analytic number theory. For example, the famous Lindelöf hypothesis for the Riemann zeta-function $\zeta(s)$ which says that, for every $\varepsilon > 0$,

$$\zeta\Big(\frac{1}{2} + it\Big) \ll_{\varepsilon} t^{\varepsilon}, \ t \geqslant t_0 > 0,$$

is equivalent to the mean value estimates

$$\frac{1}{T}\int_0^T \Big|\zeta\Big(\frac{1}{2} + it\Big)\Big|^{2k} \ll_{k,\varepsilon} T^{\varepsilon}, k \in \mathbb{N}$$

There exists a conjecture that, for all $k \geqslant 0$ and $T \to \infty$,

$$I_k(T) \stackrel{\text{def}}{=} \frac{1}{T} \int_0^T \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^{2k} dt \sim c(k)(\log T)^{k^2} \tag{3}$$

with some constant $c(k) > 0$. G.H. Hardy and J.E. Littlewood proved [5] that $c(1) = 1$, and A.E. Ingham found [7] the value $c(2) = \frac{1}{2\pi^2}$. Let $u \geqslant 0$ be bounded by a constant. Then in [10] it was obtained that $c\Big(\frac{u}{\sqrt{2\log\log T}}\Big) = 1$. Of course, the conjecture (3) is very complicated.

Also, the estimates for $I_k(T)$ are known. The first results in this direction were obtained by K. Ramachandra. For example, he proved in [17] that

$$I_{\frac{1}{2}}(T) \ll T(\log T)^{\frac{1}{2}}.$$

The further progress in the field belongs to D.R. Heath-Brown. In [6], he proved the estimate

$$I_k(T) \gg_k T(\log T)^{k^2} \tag{4}$$

for all rational $k \geqslant 0$, and the estimate

$$I_k(T) \ll_k T(\log T)^{k^2} \tag{5}$$

for $k = \frac{1}{m}$, $m \in \mathbb{N}$. Moreover, he obtained under the Riemann hypothesis (all non-trivial zeros of $\zeta(s)$ lie on the critical line) that (4) holds for all $k \geqslant 0$ and (5) is true for $0 \leqslant k \leqslant 2$. To prove this, D.R. Heath-Brown applied the Gabriel convexity theorems, see, for example, [10].

For the Estermann zeta-function, the mean square was studied in [18], see also [19].

Theorem 1. *For* $\sigma > \frac{1}{2}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_1^T \left|E\Big(\sigma + it; \frac{k}{l}, \alpha\Big)\right|^2 dt = \frac{\zeta(2\sigma - 2a)\zeta^2(2\sigma - a)\zeta(2\sigma)}{3(4\sigma - 2a)}.$$

Moreover, if $a < 0$, *then*

$$\int_1^T \left|E\Big(\sigma + it, \frac{k}{l}, \alpha\Big)\right|^2 dt \ll \begin{cases} T, & \textit{if } \sigma > \frac{1}{2}, \\ T\log^2 T, & \textit{if } \sigma = \frac{1}{2}, \\ T^{2(1-\sigma)}, & \textit{if } a + \frac{1}{2} < \sigma < \frac{1}{2}, \\ T^{1-2a}\log^2 T, & \textit{if } \sigma = a + \frac{1}{2}, \\ T^{3-4\sigma+2a}, & \textit{if } \sigma < a + \frac{1}{2}. \end{cases}$$

For the proof, a representation of $E\big(s; \frac{k}{l}, \alpha\big)$ by Dirichlet L-functions is used. Denote by $\varphi(m)$ the Euler function,

$$\varphi(m) = m \prod_{p/m} \Big(1 - \frac{1}{p}\Big),$$

by $\mu(m)$ the Möbius function,

$$\mu(m) = \begin{cases} 1, & \text{if } m = 1, \\ (-1)^r, & \text{if } m = p_1 \ldots p_r,\ p_j \text{ is prime, } j = 1, .., r, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, let

$$\tau(\chi)=\sum_{m \bmod q}\chi(m)\exp\Big\{2\pi i\frac{m}{q}\Big\}$$

be the Gauss sum associated to the Dirichlet character $\chi \bmod q$. Then, for all s,

$$E\Big(s;\frac{k}{l},\alpha\Big)=\sum_{d|l}\frac{1}{d^s\varphi\big(\frac{l}{d}\big)}\sum_{\substack{b|\frac{l}{d}\\ \left(b,\frac{l}{db}\right)=1}}\mu\Big(\frac{l}{db}\Big)\sum_{\substack{\chi(\bmod b)\\ primitive}}\tau(\overline{\chi})\chi\Big(kd\Big(\overline{\frac{l}{b}}\Big)\Big)\times$$

$$\times\lambda(s,d,\chi,\alpha)L(s,\chi)L(s-\alpha,\chi),$$

where $L(s,\chi)$ denote a Dirichlet L–function, and $\lambda(\alpha+it,m,\chi,\alpha)\ll|\sigma_\alpha(m)|$. From this it follows that

$$\int_1^T\Big|E\Big(\sigma+it;\frac{k}{l},\alpha\Big)\Big|^2dt\ll\sum_{b|l}\sum_{\substack{\chi(\bmod b)\\ primitive}}\Big(\int_1^T|L(\sigma+it,\chi)|^4dt\int_1^T|L(\sigma-\alpha+it,\chi)|^4dt\Big)^{\frac{1}{2}},$$

and to prove Theorem 1 it remains to apply the results for the fourth moment of Dirichlet L-functions.

Y. Kamiya in [8] obtained an average mean square estimate for $E\big(s;\frac{k}{l},\alpha\big)$. He proved that, for $A>49$ and $T\to\infty$,

$$\sum_{\substack{k=1\\(k,l)=1}}^{l}\int_{[-T,T]\setminus[-A,A]}\Big|E\Big(\frac{1}{2}+it;\frac{k}{l},0\Big)\Big|^2dt\ll lT\log^4 lT.$$

The latter estimate was improved in [20].

Theorem 2. *Uniformly for $l\leqslant T$ as $T\to\infty$,*

$$\frac{1}{\varphi(l)}\sum_{\substack{k=1\\(k,l)=1}}^{l}\int_1^T\Big|E\Big(\frac{1}{2}+it;\frac{k}{l},0\Big)\Big|^2dt\asymp T\log^4 T.$$

If l is prime, then

$$\sum_{k=1}^{l-1}\int_1^T\Big|E\Big(\frac{1}{2}+it;\frac{k}{l},0\Big)\Big|^2dt=\frac{l^5-l^4+7l^3-11l^2+5l+1}{2\pi^2(l-1)l^2(l+1)}T\log^4 T+O(T\log^3 T).$$

2. Zero distribution of $E\big(s;\frac{k}{l},\alpha\big)$

The zero distribution of zeta-functions is one of the most interesting problems and has numerous applications. B. Riemann was the first who observed a close relation of the Riemann zeta-function to the distribution of prime numbers. In 1896 de la Vallée Poussin and Hadamard proved independently that $\zeta(1+it)\neq 0$, and this allowed them to obtain the asymptotic law of prime numbers:

$$\pi(x)=\sum_{p\leqslant x}1\sim\int_2^x\frac{du}{\log u},\ x\to\infty.$$

As it was noted in Section 2, the famous Riemann hypothesis (RH) asserts that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$. If this hypothesis is true, then

$$\pi(x) = \int_2^x \frac{du}{\log u} + O(x^{\frac{1}{2}} \log x).$$

From the latter estimate RH also follows.
The zero distribution of the function $E\left(s; \frac{k}{l}, \alpha\right)$ depends on the parameters $\frac{k}{l}$ and α. It is not difficult to see that

$$E\left(s; \frac{k}{l}, \alpha\right) \neq 0$$

for $\sigma > 3$. The functional equation for $E\left(s; \frac{k}{l}, \alpha\right)$ shows that, for $\sigma < -2 + \operatorname{Re}\alpha$, $E\left(s; \frac{k}{l}, \alpha\right) = 0$ near the real axis. Zeros $\rho = \beta + i\gamma$ of $E\left(s; \frac{k}{l}, \alpha\right)$ in this region are called trivial. It is easily seen that

$$T \ll \sharp\{\rho \text{ is trivial} : |\rho| \leqslant T\} \ll T.$$

The non-trivial zeros of $E\left(s; \frac{k}{l}, \alpha\right)$ lie in the region $\{s \in \mathbb{C} : -2 + \operatorname{Re}\alpha \leqslant \sigma \leqslant 3\}$. Denote by $N\left(T; \frac{k}{l}, \alpha\right)$ the number of non-trivial zeros of $E\left(s; \frac{k}{l}, \alpha\right)$ with $|\gamma| \leqslant T$. Then in [21] the following asymptotic formula has been obtained.
Theorem 3. *Let* $T \to \infty$. *Then*

$$N\left(T; \frac{k}{l}, \alpha\right) = \frac{2T}{\pi} \log \frac{lT}{2\pi e} + O(\log T).$$

We see that the main term in the formula for $N\left(T; \frac{k}{l}, \alpha\right)$ does not depend on the parameters k and α.
Theorem 3 is a corollary of a general result obtained in [21]. Recall that $a = \operatorname{Re}\alpha$. Let $B > 3 - a$ be a constant, and $T \to \infty$. Then

$$\sum_{\substack{\beta > -B \\ |\gamma| \leqslant T}} (B + \beta) = (2B + a + 1)\frac{T}{\pi} \log \frac{Tl}{2\pi e} + O(\log T).$$

This and Theorem 3 imply the asymptotics for the mean value of the real parts of non-trivial zeros.
Theorem 4. [21]. *Let* $T \to \infty$. *Then*

$$N^{-1}\left(T; \frac{k}{l}, \alpha\right) \sum_{\substack{\rho \text{ non-trivial} \\ |\gamma| \leqslant T}} \beta = \frac{a+1}{2} + O(T^{-1}).$$

Theorem 4 suggests an idea that the non-trivial zeros of $E\left(s; \frac{k}{l}, \alpha\right)$ lie on the line $\sigma = \frac{a+1}{2}$. However, if RH holds, this is not true in general. Really, by the definition of $E\left(s; \frac{k}{l}, \alpha\right)$

$$E(s; 1, \alpha) = \zeta(s)\zeta(s - \alpha). \tag{6}$$

Thus, if RH holds, then $E(s; 1, \alpha) = 0$ on the lines $\sigma = \frac{1}{2}$ and $\sigma = \frac{1}{2} + a$, and $E(s; 1, \alpha) \neq 0$ on the line $\sigma = \frac{a+1}{2}$.

Denote by $N\left(\sigma, T; \frac{k}{l}, \alpha\right)$ the number of non-trivial zeros $\rho = \beta + i\gamma$ of the function $E\left(s; \frac{k}{l}, \alpha\right)$ with $\beta > \sigma$ and $|\gamma| \leqslant T$. Then in [21] the following bounds for $N\left(\sigma, T; \frac{k}{l}, \alpha\right)$ were obtained.

Theorem 5. *Let $T \to \infty$. Then uniformly in $\delta > 0$*

$$N\left(\frac{1}{2} + \delta; T, \frac{k}{l}, \alpha\right) \ll \frac{T \log\log T}{\delta} \ll \frac{\log\log T}{\delta \log T} N\left(T; \frac{k}{l}, \alpha\right),$$

and, for fixed $\sigma > \frac{1}{2}$,

$$N\left(\sigma, T; \frac{k}{l}, \alpha\right) \ll T.$$

For the proof the Littlewood theorem, the Jensen formula and the Jensen inequality on convex functions are applied.
Theorem 5 shows that the set of zeros on the right of the curve

$$\sigma = \frac{1}{2} + \psi(t)\frac{\log\log t}{\log t},$$

where $\psi(t) > 0$ and $\psi(t) \to \infty$ as $t \to \infty$, has zero density in the set of all non-trivial zeros. Example (6) leads to the following conjecture.

Conjecture. *At least a positive proportion of the non-trivial zeros of $E\left(s; \frac{k}{l}, \alpha\right)$ is clustered around the lines $\sigma = \frac{1}{2}$ and $\sigma = \frac{1}{2} + a$.*

3. Universality

In [22] S.M. Voronin obtained the universality of the Riemann zeta-function. Let $0 < r < \frac{1}{4}$, and let $f(s)$ be a continuous non-vanishing function on the disc $|s| \leqslant r$ which is analytic in the interior of this disc. Then he proved that, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s| \leqslant r} \left|\zeta\left(s + \frac{3}{4} + i\tau\right) - f(s)\right| < \varepsilon.$$

Later, S.M. Gonek, A. Reich, B. Bagchi, K. Matsumoto, J. Stending, R. Stending, W. Schwarz, R. Garunkštis, H. Mishou, J. Genys, V. Garbaliauskienė, H. Nagoshi, R. Macaitienė, the author and others improved and generalized the Voronin theorem. Define

$$\nu_T(\ldots) = \frac{1}{T} meas\{\tau \in [0, T] : \ldots\},$$

where $meas\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and in place of dots a condition satisfied by τ is to be written. The final version of the Voronin theorem is the following [10].

Theorem 6. *Let K be a compact subset of the strip $D_0 = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \nu_T\left(\sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon\right) > 0.$$

The case of the function $E\left(s;\frac{k}{l},\alpha\right)$ is more complicated, since the factor $\exp\left\{2\pi im\frac{k}{l}\right\}$ is not multiplicative. Let χ be the Dirichlet character mod l, and, for $\sigma>1$ (we recall that $a\leqslant 0$),

$$E(s;\chi,\alpha)=\sum_{m=1}^{\infty}\frac{\sigma_\alpha(m)}{m^s}\chi(m).$$

Thus, in the definition of $E\left(s;\frac{k}{l},\alpha\right)$ the arithmetic function $\exp\left\{2\pi im\frac{k}{l}\right\}$ is replaced by a multiplicative function $\chi(m)$. It turns out that $E\left(s;\frac{k}{l},\alpha\right)$ is a linear combination of the functions $E(s;\chi,\alpha)$. For simplicity, suppose that l is a prime number.

Theorem 7. [4]. *Let l be prime. Then*

$$E\left(s;\frac{k}{l},\alpha\right)=\frac{1}{\varphi(l)}\sum_{\substack{\chi(\mathrm{mod}\, l)\\ \chi\neq\chi_0}}\tau(\overline{\chi})\chi(k)E(s;\chi,\alpha)+\Lambda(s,\alpha)E(s;\chi_0,\alpha),$$

where, for $\sigma>0$,

$$\Lambda(s,\alpha)=\begin{cases}\dfrac{2l-l^{1-s}-l^s}{l^s(l-1)(1-l^{-s})^2}, & \text{if } \alpha=0,\\[2ex] \dfrac{l-l^{1+\alpha-s}-l^{1+2\alpha}+l^{1+2\alpha-s}-l^s+l^{\alpha+s}}{l^s(l-1)(1-l^\alpha)(1-l^{-s})(1-l^{\alpha-s})}, & \text{otherwise}.\end{cases}$$

The statement of Theorem 7 is also valid in the opposite direction.

Theorem 8. [4]. *Let l be prime, and χ be a character mod l. Then*

$$E(s;\chi,\alpha)=\frac{1}{\tau(\overline{\chi})}\sum_{m(\mathrm{mod}\, l)}\overline{\chi}(m)E\left(s;\frac{m}{l},\alpha\right)$$

if $\chi\neq\chi_0$, and otherwise

$$E(s;\chi_0,\alpha)=\begin{cases}(1-l^{-s})^2E(s;1,0), & \text{if } \alpha=0,\\[2ex] \dfrac{l^s-l^{s+\alpha}-1+l^{\alpha-s}+l^{2\alpha}-l^{2\alpha-s}}{l^s(1-l^\alpha)}E(s;1,\alpha), & \text{otherwise}.\end{cases}$$

In any case,

$$E(s;\chi,\alpha)=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}\left(1-\frac{\chi(p)}{p^{s-\alpha}}\right)^{-1}=L(s,\chi)L(s-\alpha,\chi);$$

the Euler product representation is valid for $\sigma>\max(1+a,1)$ while the later formula holds for all s. If $\chi\neq\chi_0$, then $E(s;\chi,\alpha)$ is an entire function. $E(s;\chi_0,\alpha)$ has simple poles at $s=1$ and $s=1+\alpha$.

The proofs of Theorems 7 and 8 are based on the following assertions. Let $(k,l)=1$. Then

$$\exp\left\{2\pi i\frac{k}{l}\right\}=\frac{1}{\varphi(l)}\sum_{\chi(\mathrm{mod}\, l)}\tau(\overline{\chi})\chi(k)$$

and

$$\tau(\overline{\chi})\chi(k) = \sum_{m(mod\, l)} \overline{\chi}(m) \exp\Big\{2\pi i \frac{mk}{l}\Big\}.$$

Moreover,

$$\exp\Big\{2\pi i \frac{k}{l}\Big\} = \frac{1}{\varphi(l)} \sum_{\substack{m|l \\ \left(m, \frac{l}{m}\right) = 1}} \mu\Big(\frac{m}{l}\Big) \sum_{\substack{\chi(mod\, m) \\ primitive}} \tau(\overline{\chi})\chi\Big(k\overline{\Big(\frac{l}{m}\Big)}\Big).$$

Since the function $E(s;\chi,\alpha)$ has the Euler product, a joint universality for it can be proved. Note that the first joint universality theorem for Dirichlet L–functions with pairwise non-equivalent characters was obtained by S.M Voronin in [23].

Theorem 9. [4]. *Suppose that* $a < -1$, $l \geqslant 5$ *is prime, and that, for* $p = 2, 3$,

$$\sum_{m=1}^{\infty} \frac{|\sigma_\alpha(p^m)|}{p^{m\beta}} < 1 \tag{7}$$

with some $\beta \in \left(\frac{1}{2}, 1\right)$. *For* $1 \leqslant j \leqslant \varphi(l)$, *let* χ_j *be a Dirichlet character mod l,* K_j *be a compact subset of the strip* $D_\beta = \{s \in \mathbb{C} : \beta < \sigma < 1\}$ *with connected complement, and let* $g_j(s)$ *be a continuous non-vanishing function on* K_j *which is analytic in the interior of* K_j. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\big(\sup_{1\leqslant j\leqslant\varphi(l)} \sup_{s\in K_j} |E(s+i\tau;\chi_j,\alpha) - g_j(s)| < \varepsilon\big) > 0.$$

Now Theorems 7 and 9 imply the universality of the Estermann zeta-function.

Theorem 10. [10]. *Suppose that* $k \neq 1$, $l \neq 1$, $a < -1$, $l \geqslant 5$ *is prime, and that, for* $p = 2, 3$, *(7) holds. Let* K *be a compact subset of the strip* D_β *with connected complement, and let* $f(s)$ *be a continuous function on* K *which is analytic in the interior of* K. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{s\in K}\Big|E\Big(s+i\tau;\frac{k}{l},\alpha\Big) - f(s)\Big| < \varepsilon\Big) > 0.$$

Note that in Theorem 10, differently from Theorem 6, the function $f(s)$ *is not necessarily non-vanishing on* K. *This difference is explained by the existence of the Euler product for* $\zeta(s)$ *while, for* $k \neq 1$, $l \neq 1$, *the function* $E\big(s;\frac{k}{l},\alpha\big)$ *has not this product.*

Theorem 10 gives some information on the zero distribution of the function $E\big(s;\frac{k}{l},\alpha\big)$.

Corollary. *Under the assumptions of Theorem 10, for fixed* $\sigma \in (\beta, 1)$ *and* $T \to \infty$,

$$T \ll N\Big(\sigma, T; \frac{k}{l}, \alpha\Big) \ll T.$$

Moreover, the real parts of zeros of the function $E\big(s;\frac{k}{l},\alpha\big)$ *lie dense in the interval* $(\beta, 1)$.

In Theorem 10, the number l is prime. However, we conjecture that the function $E\big(s;\frac{k}{l},\alpha\big)$ is universal in the Voronin sense for all l.

4. Limit theorems

The first probabilistic result for zeta-functions was obtained by H. Bohr and B. Jessen. Let R be any closed rectangle on the complex plane with the edges parallel to the axes, and let $L_\sigma(T,R)$ denote the Jordan measure of the set

$$\{t \in [0,T] : \log\zeta(\sigma+it) \in R\}.$$

Suppose that $\sigma > 1$. Then in [1] they proved that there exists the limit

$$\lim_{T\to\infty} \frac{L_\sigma(T,R)}{T} = W_\sigma(R)$$

which depends only on σ and R. In [2] an analogous result was obtained for $\sigma > \frac{1}{2}$. Let

$$G = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\} \setminus \bigcup_{s_j=\sigma_j+it_j} \{s = \sigma + it_j : \frac{1}{2} < \sigma < \sigma_j\},$$

where s_j runs through all zeros of $\zeta(s)$ in the region $\frac{1}{2} < \sigma < 1$. Denote by $L_{1,\sigma}(T,R)$ the Jordan measure of the set

$$\{t \in [0,T] : \sigma + it \in G, \log\zeta(\sigma+it) \in R\}.$$

Then in [2] H. Bohr and B. Jessen proved that there exists the limit

$$\lim_{T\to\infty} \frac{L_{1,\sigma}(T,R)}{T} = W_{1,\sigma}(R)$$

which depends only on σ and R. For the proof of the above results the theory of sums of convex curves was used.

K. Matsumoto estimated [15], [16] the rate of convergence in Bohr-Jessen's theorems.

Bohr-Jessen's ideas were developed by A. Wintner, V. Borchsenius, A. Selberg, P.D. T.A. Elliott, A. Ghosh, B. Bagchi, K. Matsumoto, J. Steuding, W. Schwarz, R. Kačinskaitė, R. Šleževičienė-Steuding, J. Genys, R. Macaitienė, V. Garbaliauskienė, the author and others.

The modern version of Bohr-Jessen's results can be stated in the following form. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S, and let P_n and P, $n \in \mathbb{N}$, be probability measures on $(S, \mathcal{B}(S))$. We recall that P_n converges weakly to P as $n \to \infty$ if, for every real continuous bounded function f on S,

$$\lim_{n\to\infty} \int_S f dP_n = \int_S f dP.$$

Theorem 11. [10]. *Suppose that $\sigma > \frac{1}{2}$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the probability measure*

$$\nu_T(\zeta(\sigma+it) \in A),\ A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $T \to \infty$.

Note that the explicit form of the limit measure P_σ can be given. Now let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ denote the unit circle on the complex plane, and

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime p. By the Tikhonov theorem, with the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined, and this gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and put, for $m \in \mathbb{N}$,

$$\omega(p) = \sum_{p^r \| m} \omega^r(p),$$

where $p^r \| m$ means that $p^r | m$ but $p^{r+1} \nmid m$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define, for $\sigma > \frac{1}{2}$, the complex-valued random element $E\left(\sigma; \frac{k}{l}, \alpha, \omega\right)$ by

$$E\left(\sigma; \frac{k}{l}, \alpha, \omega\right) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)\omega(m)}{m^\sigma} \exp\left\{2\pi i m \frac{k}{l}\right\},$$

and denote by $P_{E,\sigma}^{\mathbb{C}}$ its distribution, i.e.,

$$P_{E,\sigma}^{\mathbb{C}}(A) = m_H\left(\omega \in \Omega : E\left(\sigma; \frac{k}{l}, \alpha, \omega\right) \in A\right), \ A \in \mathcal{B}(\mathbb{C}).$$

Theorem 12. [12]. *Suppose that* $\sigma > \frac{1}{2}$, $a \leqslant 0$ *and* $k \neq 1$, $l \neq 1$. *Then the probability measure*

$$\nu_T\left(E\left(\sigma + i\tau; \frac{k}{l}, \alpha\right) \in A\right), \ A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{E,\sigma}^{\mathbb{C}}$ *as* $T \to \infty$.

Theorem 12 admits a joint generalization. Let, for $\sigma > \max(1, 1 + \operatorname{Re}\alpha_j)$, $(k_j, l_j) = 1$,

$$E\left(s; \frac{k_j}{l_j}, \alpha_j\right) = \sum_{m=1}^{\infty} \frac{\sigma_{\alpha_j}(m)}{m^s} \exp\left\{2\pi i m \frac{k_j}{l_j}\right\}, \ j = 1, ..., r.$$

Denote

$$\mathbb{C}^r = \underbrace{\mathbb{C} \times ... \times \mathbb{C}}_{r}.$$

Suppose that $a_j \leqslant 0$, $j = 1, ..., r$, and for $\min_{1 \leqslant j \leqslant r} \sigma_j > \frac{1}{2}$ and $\omega \in \Omega$, define

$$E(\sigma_1, ..., \sigma_r; \omega) = \left(E\left(\sigma_1; \frac{k_1}{l_1}, \alpha_1, \omega\right), ..., E\left(\sigma_r; \frac{k_r}{l_r}, \alpha_r, \omega\right)\right),$$

where

$$E\left(\sigma_j; \frac{k_j}{l_j}, \alpha_j, \omega\right) = \sum_{m=1}^{\infty} \frac{\sigma_{\alpha_j}(m)\omega(m)}{m^{\sigma_j}} \exp\left\{2\pi i m \frac{k_j}{l_j}\right\}, \ j = 1, .., r.$$

Theorem 13. [14]. *Suppose that* $\min_{1\leqslant j\leqslant r}\sigma_j > \frac{1}{2}$, $a_j \leqslant 0$ *and* $k_j \neq 1$, $l_j \neq 1$, $j = 1, \dots, r$. *Then the probability measure*

$$\nu_T\Big(\Big(E\Big(\sigma_1 + i\tau; \frac{k_1}{l_1}, \alpha_1\Big), \dots, E\Big(\alpha_r + i\tau; \frac{k_r}{l_r}, \alpha_r\Big)\Big) \in A\Big),\ A \in \mathcal{B}(\mathbb{C}^r),$$

converges weakly to the distribution of the random element $E(\sigma_1, \dots, \sigma_r; \omega)$ *as* $T \to \infty$.

Another generalization of Theorem 12 is a limit theorem in the space of meromorphic functions. Let $D_1 = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$, and let $M(D_1)$ denote the space of meromorphic on D_1 functions equipped with the topology of uniform convergence on compacta. Moreover, $H(D_1)$ is the space of analytic on D_1 functions with the same topology. $H(D_1)$ is a subspace of $M(D_1)$.
On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define the $H(D_1)$-valued random element

$$E\Big(s; \frac{k}{l}, \alpha, \omega\Big) = \sum_{m=1}^{\infty} \frac{\sigma_\alpha(m)\omega(m)}{m^s} \exp\Big\{2\pi i m \frac{k}{l}\Big\},$$

and let

$$P_{E,H}(A) = m_H\Big(\omega \in \Omega : E\Big(s; \frac{k}{l}, \alpha, \omega\Big) \in A\Big),\ A \in \mathcal{B}(H(D)),$$

be its distribution. Then we have the following result [13].

Theorem 14. *Suppose that* $a \leqslant 0$ *and* $k \neq 1$, $l \neq 1$. *Then the probability measure*

$$\nu_T\Big(E\Big(s + i\tau; \frac{k}{l}, \alpha\Big) \in A\Big),\ A \in \mathcal{B}(M(D_1)),$$

converges weakly to $P_{E,H}$ *as* $T \to \infty$.

A joint version of Theorem 14 also can be obtained.

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НЕКОТОРЫЕ ТЕОРЕМЫ О РАСПРЕДЕЛЕНИИ ЗНАЧЕНИЙ ДЛЯ ДЗЕТА-ФУНКЦИЙ ЭСТЕРМАНА

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В работе представлен обзор следующих результатов: оценки главных значений, распределение нулей, универсальные и предельные теоремы в смысле слабой сходимости вероятностных мер дзета-функции Эстерманна.

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