VALUE DISTRIBUTION OF GENERAL DIRICHLET SERIES

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In the paper a survey of some probabilistic results and their application to universality for general Dirichlet series is given.

Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all positive integers, integers, real and complex numbers, respectively. Let $\{a_m\}$ be a sequence of complex numbers, and let $\{\lambda_m\}$ be an increasing sequence of positive numbers such that $\lim\limits_{m\to\infty} \lambda_m = +\infty$. Denote by $s = \sigma + it$ a complex variable. Then the series

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s} \tag{1}$$

is called a general Dirichlet series with coefficients a_m and exponents λ_m. If $\lambda_m = \log m$, then the series (1) becomes the ordinary Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

Dirichlet series play an important role in number theory, however, their value distribution is very complicated. In many cases, difficult problems of individual values can be reduced to various average results. Among them the probabilistic methods in the investigations of Dirichlet series belongs to a famous mathematician H. Bohr. In the second decade of the last century he proposed a statistical approach in the theory of the Riemann zeta-function. H. Bohr implemented his idea in joint works with B. Jessen, [3], [4]. Later A. Wintner, V. Borchsenius, A. Selberg, P.D. T.A. Elliott, A. Ghosh, E. Stankus, B. Bagchi, D. Joyner, A. Laurinčikas, E.M. Nikishin, K. Matsumoto, R. Garunkštis, J. Steuding and others continued the investigations of H. Bohr and B. Jessen and obtained modern probabilistic results in this field. However, the majority of their investigations were related to ordinary Dirichlet series. On the other hand, the known limit theorems for general Dirichlet series were

[1]Genys Jonas (genys@splius.lt), Dept. of Mathematics and Informatics, Siauliai University, P. Višinskio St. 19, LT-77156 Šiauliai, Lithuania.

weaker than similar theorems for ordinary Dirichlet series. All these circumstances suggested an idea of the necessity to improve the results for general Dirichlet series.

It is well known that Dirichlet series are related to almost periodic functions: almost periodic functions are presented by Dirichlet series. Therefore, we recall some results for almost periodic functions. Denote by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

A number τ is called the ε-almost period of the function $g(t)$, $-\infty < t < +\infty$, if

$$|g(t+\tau) - g(t)| < \varepsilon$$

for all $t \in \mathbb{R}$. A set $E \subset \mathbb{R}$ is called relatively dense if there exists a number $l > 0$ such that every interval of length l contains at least one number of the set E.

A continuous on $\mathbb{R}$ function $g(t)$ is called almost periodic in the Bohr sense if for every $\varepsilon > 0$ there exists a relatively dense set of ε-almost periods of the function $g(t)$.

A. Wintner proved [29] that the distribution functions

$$\frac{1}{2T}\text{meas}\{t \in [-T, T]: \ \Re g(t) < x\} \tag{2}$$

and

$$\frac{1}{2T}\text{meas}\{t \in [-T, T]: \ \Im g(t) < x\}, \tag{3}$$

where $g(t)$ is an almost periodic in the Bohr sense function, converge to distribution functions at their continuity points as $T \to \infty$.

Also, almost periodic functions in the Besicovitch sense have similar statistical properties. Let $p \geqslant 1$ and, for $g_1(t), g_2(t) \in L_p$,

$$\varrho_{B_p}(g_1, g_2) = \limsup_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T}|g_1(t) - g_2(t)|^p \mathrm{d}t\right)^{\frac{1}{p}}.$$

A function $g(t)$ is called B_p-almost periodic, shortly $g \in B_p$, if there exists a trigonometric polynomial

$$p_n(t) = \sum_{m=1}^{n} b_m \mathrm{e}^{i\lambda_m t}$$

such that

$$\lim_{n\to\infty} \varrho_{B_p}(g, p_n) = 0.$$

E.M. Nikishin proved [25] that if $g(t) \in B_1$, then the distribution functions (2) and (3) also converge to some distribution functions at their continuity points as $T \to \infty$. Since the class B_1 is the widest one, therefore the functions $g \in B_p$, $p > 1$, also have the above property.

1. Limit theorems in the space of meromorphic functions

It is well known, see for example, [24], that the Dirichlet series (1) absolutely converges in half-plane $D_a = \{s \in \mathbb{C} : \sigma > \sigma_a\}$. Denote its sum by $f(s)$. Then $f(s)$ is a holomorphic functions in D_a. Therefore, it is reasonable to consider how often the function $f(s + i\tau)$ belongs to some given set of analytic functions. Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Let

$$\nu_T(\ldots) = \frac{1}{T}\mathrm{meas}\{\tau \in [0, T] : \ \ldots\},$$

where in place of dots a conditions satisfied by τ is to be written. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S. Then in [12] the following statement has been proved.

Theorem 1.1. *On $(H(D_a), \mathcal{B}(H(D_a)))$ there exists a probability measure P such that the measure*

$$\nu_T(f(s + i\tau) \in A), \quad A \in \mathcal{B}(H(D_a)),$$

converges weakly to P as $T \to \infty$.

When the system of exponents $\{\lambda_m\}$ is linearly independent over the field of rational numbers, the limit measure P in Theorem 1 can be explicitly defined. This needs one topological structure.

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane, and

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. With the product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H exists, and this gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m, $m \in \mathbb{N}$ and define on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ an $H(D_a)$-valued random element $f(s, \omega)$ by

$$f(s, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) \mathrm{e}^{-\lambda_m s}.$$

Let P_f be the distribution of the random element $f(s, \omega)$, i.e.,

$$P_f(A) = m_H(\omega \in \Omega : \ f(s, \omega) \in A), \quad A \in \mathcal{B}(H(D_a)).$$

Then in [12] the following theorem was obtained.

Theorem 1.2. *If the system of exponent* $\{\lambda_m\}$ *is linearly independent over the field of rational numbers then the probability measure*

$$\nu_T(f(s+i\tau)\in A), \quad A\in\mathcal{B}(H(D_a)),$$

converges weakly to the measure P_f *as* $T\to\infty$.

Now suppose that the function $f(s)$ is meromorphically continuable to the half-plane $\sigma>\sigma_1$, $\sigma_1<\sigma_a$, and that all poles in this region are in a compact set. Moreover, let, for $\sigma>\sigma_1$, the estimates

$$f(\sigma+it)=O(|t|^{\alpha}), \quad |t|\geqslant t_0, \quad \alpha>0, \tag{4}$$

and

$$\int_{-T}^{T}|f(\sigma+it)|^2 dt=O(T), \quad T\to\infty, \tag{5}$$

be valid. Also,

$$\lambda_m\geqslant c(\log m)^{\delta} \tag{6}$$

with some positive constants c and δ. Let $D_1=\{s\in\mathbb{C}:\sigma>\sigma_1\}$. All these conditions implies that $f(s,\omega)$ is an $H(D_1)$-valued random element defined on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. To prove this, the pairwise orthogonality of random variables $\omega(m)$, $m\in\mathbb{N}$, and the Radamacher theorem [23] are used.

Denote by $M(D_1)$ the space of meromorhpic on D_1 functions equipped with the topology of uniform convergence on compacta.

Theorem 1.3. [13] *Suppose that the function* $f(s)$ *satisfies conditions (4) and (5). Then on* $(M(D_1),\mathcal{B}(M(D_1)))$ *there exists a probability measure* P *such that the measure*

$$\nu_T(f(s+i\tau)\in A), \quad A\in\mathcal{B}(M(D_1)),$$

converges weakly to the P *as* $T\to\infty$.

The first attempt to identify the limit measure P in Theorem C was made in [20], and the following result has been obtained.

Theorem 1.4. *Suppose that the set* $\{\log 2\}\cup\bigcup_{m=1}^{\infty}\{\lambda_m\}$ *is linearly independent over the field of rational numbers, and that conditions (4) – (6) are satisfied. Then the limit measure* P *in Theorem 1.3 coincides with the distribution of* $H(D_1)$*–valued random element* $f(s,\omega)$.

Of course, the presence of the number $\log 2$ in the statement of Theorem 1.4 is not natural. A new method of the proof gave the following result [5].

Theorem 1.5. *Suppose that the system of exponents* $\{\lambda_m\}$ *is linearly independent over the field of rational numbers, and that conditions (4)–(6) are satisfied. Then the probability measure*

$$\nu_T(f(s+i\tau)\in A), \quad A\in\mathcal{B}(M(D_1)),$$

converges weakly to the distribution of the $H(D_1)$*-valued random element* $f(s,\omega)$.

2. Joint limit theorems

Let, for $\sigma > \sigma_{aj}$,

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_{mj} s}$$

with $a_{mj} \in \mathbb{C}$ and $\lambda_{mj} \in \mathbb{R}$, $\lambda_{1j} < \lambda_{2j} < \ldots$, $\lim\limits_{m\to\infty} \lambda_{mj} = +\infty$, $j = 1, \ldots, n$. We assume that the functions $f_1(s), \ldots, f_n(s)$ are meromorphically continuable to the half-planes $\sigma > \sigma_{11}$, $\sigma_{11} < \sigma_{a1}, \ldots, \sigma > \sigma_{1n}$, $\sigma_{1n} < \sigma_{an}$, respectively, and all poles in these regions are included in a compact sets. Also, we suppose that, for $\sigma > \sigma_{1j}$, the estimates

$$f_j(s) = O(|t|^{\alpha_j}), \quad |t| \geqslant t_0, \quad \alpha_j > 0, \tag{7}$$

and

$$\int_{-T}^{T} |f_j(\sigma + it)|^2 \mathrm{d}t = O(T), \quad T \in \infty, \tag{8}$$

hold, $j = 1, \ldots, n$. Moreover, we assume that

$$\lambda_{mj} \geqslant c_j (\log m)^{\theta_j} \tag{9}$$

with some positive constants c_j and θ_j, $j = 1, \ldots, n$.

Let $D_j = \{s \in \mathbb{C} : \sigma > \sigma_{1j}\}$, $j = 1, \ldots, n$, and

$$H_n = H(D_1) \times \ldots \times H(D_n),$$

$$M_n = M(D_1) \times \ldots \times M(D_n).$$

Denote

$$\hat{\Omega} = \Omega_1 \times \ldots \times \Omega_n,$$

where $\Omega_j = \Omega$ for $j = 1, \ldots, n$, and let $\widehat{\omega} = (\omega_1, \ldots, \omega_n)$ for $\widehat{\omega} = \widehat{\Omega}$ and $\omega_j \in \Omega_j$, $j = 1, \ldots, n$. Then $\widehat{\Omega}$ is also a compact topological group, and on $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}))$ the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$, $j = 1, \ldots, n$.

The first joint limit theorem in the space of meromorphical functions for general Dirichlet series was obtained in [22]. Define on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ an H_n-valued random element $F(s_1, \ldots, s_n; \omega)$ by the formula

$$F(s_1, \ldots, s_n; \omega) = (f_1(s_1, \omega_1), \ldots, f_n(s_n, \omega_n)),$$

where

$$f_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_{mj} s_j}, \quad s_j \in D_j, \quad j = 1, \ldots, n.$$

The main result of [22] is contained in the following statement.

Theorem 2.1. *For $j = 1, ..., n$, suppose that the sets $\{\log 2\} \cup \bigcup\limits_{m=1}^{\infty} \{\lambda_{mj}\}$ are linearly independent over the field of rational numbers, and that for $f_j(s)$ conditions (7)–(9) are satisfied. Then the probability measure*

$$P_T(A) \stackrel{def}{=} \nu_T((f_1(s + i\tau), ..., f_n(s + i\tau) \in A), \quad A \in \mathcal{B}(M_n),$$

converges weakly to the distribution of random element $F(s_1, ..., s_n; \omega)$ as $T \to \infty$.

However, the proof of Theorem 2.1 has a gap. For its validity the hypothesis on the linear independence of the $\{\log 2\} \bigcup \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$, $j = 1, ..., n$, must be replaced by that on the independence of the set $\{\log 2\} \bigcup \bigcup_{j=1}^{n} \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$. However, the main shortcoming of Theorem 2.1 is the presence of the number $\log 2$ in this hypothesis.

First the consider the case $\lambda_{mj} = \lambda_m$, $j = 1, \dots, n$. Let, for $\sigma > \sigma_{aj}$,

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_m s}, \quad j = 1, \dots, n,$$

and let in the definition of the random element $F(s_1, \dots, s_n; \omega)$

$$f_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_m s_j}, \quad s_j \in D_j, \quad j = 1, \dots, n.$$

Theorem 2.2 [6]. *Suppose that the system of exponents $\{\lambda_m\}$ is linearly independent over the field of rational numbers, and that for $f_j(s)$, $j = 1, ..., n$, conditions (7)–(9) are satisfied. Then the assertion of Theorem 2.1 is valid.*

The main result of this Section is a joint limit theorem in the space M_n with different systems of exponents. To state it we need some additional notation. Define on the probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \widehat{m}_H)$ an H_n-valued random element $F(s_1, ..., s_n; \widehat{\omega})$ by the formula

$$F(s_1, ..., s_n; \widehat{\omega}) = (f_1(s_1, \omega_1), ..., f_n(s_1, \omega_n)),$$

where

$$f_j(s_j, \omega_j) = \sum_{m=1}^{\infty} a_{mj} \omega_j(m) e^{-\lambda_{mj} s_j}, \quad s_j \in D_j,$$

and $\omega_j(m)$ is the projection of $\omega_j \in \Omega_j$ to the coordinate space γ_m, $j = 1, ..., n$. We have that the distribution P_F of the random element $F(s_1, \dots, s_n; \widehat{\omega})$ is defined by

$$P_F(A) = \widehat{m}_H((\omega_1, \dots, \omega_n) \in \widehat{\Omega} : \ F(s_1, \dots, s_n; \widehat{\omega}) \in A), \quad A \in \mathcal{B}(H_n).$$

Theorem 2.3 [7]. *Suppose that the set $\bigcup_{j=1}^{n} \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$ is linearly independent over the field of rational numbers, and that for $f_j(s)$, $j = 1, \dots, n$, conditions (7)–(9) are satisfied. Then the probability measure P_T converges weakly to P_F as $T \to \infty$.*

We note that Theorems 1.5, 2.2 and 2.3 are not empty. For example, the conditions of these theorems are satisfied by Lerch zeta-function $L(s, \lambda, \alpha)$ [16] for $\sigma > 1$, given by

$$L(s, \lambda, \alpha) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda_m}}{(m + \alpha)^s},$$

and by analytic continuation elsewhere, with transcendental α, $0 < \alpha \leqslant 1$, and real λ. In this case $a_m = e^{2\pi i\lambda_m}$ and $\lambda_m = \log(m + \alpha)$. When $\lambda \in \mathbb{Z}$, the function $L(s, \lambda, \alpha)$ reduces to the Hurwitz zeta-function and has a simple pole at $s = 1$ with residue 1. If $\lambda \notin \mathbb{Z}$, then $L(s, \lambda, \alpha)$ is an entire function.

3. The joint universality

Note that the problem of universality for zeta-functions, and, in general, for Dirichlet series comes back to S.M. Voronin. In 1975 he proved [27] that, roughly speaking, every analytic function can be approximated by translations $\zeta(s + i\tau)$ of the Riemann zeta-function $\zeta(s)$. More precisely, the last version of the Voronin theorem is the following statement [10].

Theorem 3.1. *Let K be a compact subset of the strip $\{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complement, and let $g(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T(\sup_{s\in K} |\zeta(s + i\tau) - g(s)| < \varepsilon) > 0.$$

Theorem 3.1 was generalized by many authors for other zeta-functions. For references see a survey [15].

The universality of general Dirichlet series was obtained in [21]. To state a theorem from [21] we need some additional conditions. We suppose that the system of exponents $\{\lambda_m\}$ or series (1) is linearly independent over the field of rational numbers, the function $f(s)$ is meromorphically continuable to the half-plane $\sigma > \sigma_1$ with some $\sigma_1 < \sigma$ and it is analytic in the strip

$$D = \{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_a\}.$$

We also require that the estimates (4) and (5) should be satisfied. Denote, for $x > 0$,

$$r(x) = \sum_{\lambda_m \leqslant x} 1,$$

and let $c_m = a_m e^{-\lambda_m \sigma_a}$. Suppose that, for some $\theta > 0$,

$$\sum_{\lambda_m \leqslant x} |c_m| = \theta_r(x)(1 + o(1))$$

as $x \to \infty$, $|c_m| \leqslant d$ with some $d > 0$, and

$$r(x) = C_1 x^{\varkappa} + B, \tag{10}$$

where $\varkappa \geqslant 1$, $C_1 > 0$ and $|B| \leqslant C_2$. Finally, we assume that $f(s)$ cannot be represented in the region $\sigma > \sigma_a$ by an Euler product over primes. Then in [21] the following statement has been proved.

Theorem 3.2. *Suppose that the function $f(s)$ satisfies all the conditions stated above. Let K be a compact subset of the strip D with connected complement, and let $g(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T(\sup_{s\in K} |f(s+i\tau) - g(s)| < \varepsilon) > 0.$$

The first results on the joint universality were obtained by S.M. Voronin [28], S.M. Gonek [9] and B. Bagchi [1, 2]. Independently they proved the joint universality for Dirichlet L–functions. We state a joint Voronin's theorem [28].

Theorem 3.3. *Let $0 < r < \frac{1}{4}$, and let $\chi_1, ..., \chi_n$ be pairwise non-equivalent Dirichlet characters, $g_1(s), ..., g_n(s)$ be continuous and non-vanishing on the disc $|s| \leqslant r$ and analytic in the interior of this disc. Then, for every $\varepsilon > 0$, there exists a real number τ such that*

$$\max_{|s|\leqslant r} \left| L\left(s + \frac{3}{4} + i\tau, \chi_j\right) - g_j(s)\right| < \varepsilon$$

holds for $1 \leqslant j \leqslant n$.

The joint universality for Matsumoto and Lerch zeta-functions was obtained in [11] and [17], respectively. In [18] the joint universality for zeta-functions attached to certain cusp forms was discussed, and in [26] and [19] joint universality theorems were proved for twisted Dirichlet series with multiplicative coefficient and automorphic L-functions, respectively.The joint universality for a collection general Dirichlet series with the same system of exponents was proved in [14]. The number $\log 2$ is involved in the hypothesis of [14]. As above, suppose that, for $\sigma > \sigma_{aj}$, the series

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} \mathrm{e}^{-\lambda_m s}$$

converges absolutely, $f_j(s)$ is meromorphically continuable to the half-plane $\sigma > \sigma_{1j}$ with some $\sigma_{1j} < \sigma_{aj}$, all poles being included in a compact set, it is analytic in the strip $D_j = \{s \in \mathbb{C} : \ \sigma_{1j} < \sigma < \sigma_{aj}\}$, and that $f_j(s)$ cannot be represented by an Euler product over primes in the region $\sigma > \sigma_{aj}$, $j = 1, \dots, n$. Moreover, let, for $\sigma > \sigma_{1j}$, the estimates (7) and (8) be satisfied. Let $c_{mj} = a_{mj}\mathrm{e}^{-\lambda_m \sigma_{aj}}$, $j = 1, \dots, n$. Additionally, we suppose that there exists $r \geqslant n$ sets $\mathbb{N}_k$, $\mathbb{N}_{k_1} \cap \mathbb{N}_{k_2} = \emptyset$, for $k_1 \neq k_2$, $\mathbb{N} = \bigcup_{k=1}^{r} \mathbb{N}_k$, such that $c_{mj} = b_{kj}$ for $m \in \mathbb{N}_k$, $k = 1, \dots, r$, $j = 1, \dots, n$. Let

$$L = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{r1} & \dots & b_{rn} \end{pmatrix}.$$

and we assume that the sequence $\{\lambda_m\}$ satisfies (10), and that

$$\sum_{\lambda_m \leqslant x, \ \ m\in\mathbb{N}_k} 1 = \varkappa_k r(x)(1 + o(1)), \quad x \to \infty, \tag{11}$$

with positive $\varkappa_k$, $k = 1, \dots, r$. Then in [14] the following assertion has been obtained.

Theorem 3.4. *Suppose that conditions (7), (8), (10), (11) are satisfied, the set* $\{\log 2\} \bigcup \bigcup_{m=1}^{\infty} \{\lambda_m\}$ *is linearly independent over the field of rational numbers, and that* $\operatorname{rank}(L) = n$*. Let* K_j *be a compact subset of strip* D_j *with connected complement, and let* $g_j(s)$ *be continuous function on* K_j *which is analytic in the interior of* K_j*,* $j = 1, \dots, n$*. Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\Big(\sup_{1\leqslant j\leqslant n}\sup_{s\in K_j} |f_j(s+i\tau) - g_j(s)| < \varepsilon\Big) > 0.$$

The number $\log 2$ from the conditions of joint universality theorem has been removed in the following statement [8].

Theorem 3.5. *Suppose that conditions (7), (8), (10), (11) are satisfied, the system* $\{\lambda_m\}$ *is linearly independent over the field of rational numbers, and that rank*$(L) = n$*. Then the assertion of Theorem 3.4 is true.*

The proof of Theorem 3.5 is based on Theorem 2.2.

We will give an example. Let $\{a_{mj}\}$ be a periodic sequence with period $r \geqslant n$, $j = 1, \dots, n$, and let $\lambda_m = (m+\alpha)^\beta$, where α is a transcendental number and $\beta \in (0,1)$. Then the series

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} e^{-(m+\alpha)^\beta s}$$

converges absolutely for $\sigma > \sigma_{aj} = 0$, $j = 1, \dots, n$, and $c_{mj} = a_{mj}$ is constant on the set

$$\mathbb{N}_k = \{m \in \mathbb{N} : \ m \equiv k (\mathrm{mod}\, r)\}.$$

Moreover,

$$r(x) = \sum_{(m+\alpha)^\beta \leqslant x} 1 = x^{\frac{1}{\beta}} + O(1).$$

Since α is transcendental, the system of exponents $\{\lambda_m\}$ is linearly independent over the field of rational number. Clearly, the numbers b_{kj} can be chosen in the manner such that $\operatorname{rank}(L) = n$. Therefore, if we suppose that the function $f_j(s)$ is meromorphically continuable to the region $\sigma > \sigma_{1j}$, $\sigma_{1j} < \sigma_{aj}$, $j = 1, \dots, n$, and satisfies the estimates (7) and (8), then the collection of the functions $f_1(s), \dots, f_n(s)$ is universal.

References

[1] Bagchi, B. The Statistical Behaviour and universality Properties of the Riemann Zeta-Function and other allied Dirichlet Series: Ph.D. Thesis / B. Bagchi. – Calcutta: Indian Statistical Institute, 1981.

[2] Bagchi, B. A joint universality theorem for Dirichlet L-functions / B. Bagchi // Math. Z. – 1982. – 181. – P. 319–334.

[3] Bohr, H. Über die Werverteilung der Riemanns-chen Zeta Funktion, Erste Mitteilung / H. Bohr, B. Jessen // Acta Math. – 1930. – 54.

[4] Bohr, H. Über die Werverteilung der Riemanns-chen Zeta Funktion, Zweite Mitteilung / H. Bohr, B. Jessen // Acta Math. – 58. – 1932.

[5] Genys, J. Value distribution of general Dirichlet series. IV / J. Genys, A. Laurinčikas // Lith. Math. J. – 2003. – 43(3). – P. 342–358.

[6] Genys, J. On joint value distribution of general Dirichlet series / J. Genys, A. Laurinčikas // Nonlinear Analysis: Modeling and Control, 8(2) 2003, P. 27–29.

[7] Genys, J. A joint limit theorem for general Dirichlet series / J. Genys, A. Laurinčikas // Lith. Math. J. – 2004. – 44(1). – P. 18–35.

[8] Genys, J. Value distribution of general Dirichlet series. V / J. Genys, A. Laurinčikas // Lith. Math. J. – 2004. – 44(2). – P. 144–156.

[9] Gonek, S.M. Analytic Properties of Zeta and L-Functions: Ph.D. Thesis / S.M. Gonek. – University of Michigan, 1979.

[10] Laurinčikas, A. Limit Theorems for the Riemann Zeta-Function / A. Laurinčikas. – Kluwer, Dordrecht, Boston, London, 1996.

[11] Laurinčikas, A. On the zeros of linear combinations of Matsumoto zeta-functions / A. Laurinčikas // Lith. Math. J. – 1998. – 38(2). – P. 144–159.

[12] Laurinčikas, A. Value distribution of general Dirichlet series / A. Laurinčikas // Probab. Theory and Math. Statist., Proc. 7th Vilnius Conf. B. Grigelionis at al (EDS), VSP/TEV, Utrecht/ Vilnius, 1999. – P. 405–414.

[13] Laurinčikas, A. Value distribution of general Dirichlet series. II / A. Laurinčikas // Lith. Math. J. – 2001. – 41(4). – P. 351–360.

[14] Laurinčikas, A. The joint universality for general Dirichlet series /A. Laurinčikas // Ann. Univ. Sc. Budapest, Sect. Comp. – 2003. – 2. – P. 235–251.

[15] Laurinčikas, A. The universality of zeta-functions / A. Laurinčikas // Acta Appl. Math. – 2003. – 78. – 251-271.

[16] Laurinčikas, A. The Lerch Zeta-Function / A. Laurinčikas, R. Garunkštis // Kluwer, Dordrecht, Boston, London, 2002.

[17] Laurinčikas, A. The joint universality and the functional independence for Lerch zeta-functions / // Nagoya Math. J. – 2000. – 157. – P. 211–227.

[18] Laurinčikas, A. The joint universality of zeta-functions attached to certain cusp forms / A. Laurinčikas, K. Matsumoto // Proc. Sci. Seminar Faculty of Phys. and Math., Šiauliai University, 2002. – 5. – P. 57–75.

[19] Laurinčikas, A. The joint universality of twisted automorphic L-functions / A. Laurinčikas, K. Matsumoto // J. Math. Soc. Japan. – 2004. – 56(3). – P. 923–939.

[20] Laurinčikas, A. Value distribution of general Dirichlet series. III / A. Laurinčikas, W. Schwarz, J. Steuding // Analytic and Probab. Methods in Number Theory , Proc. the Third Palanga Conf., A. Dubickas at al (Eds), TEV 2002. – P. 137–156.

[21] Laurinčikas, A. The universality of general Dirichlet series / A. Laurinčikas, W. Schwarz, J. Steuding // Analysis. – 2003. – 23. – P. 13–26.

[22] Laurinčikas, A. On joint distribution of general Dirichlet series / A. Laurinčikas, J. Steuding // Lith. Math. J. – 2002. – 42(2). – P. 163–173.

[23] Loè ve, A.F. Probability Theory / A.F. Loè ve L.;Moscow, 1968 (in Russian).

[24] Mandelbrojt, S. Séries de Dirichlet / S. Mandelbrojt. – Paris: Gauthier-Villars, 1969.

[25] Nikishin, E.M. Dirichlet series with independent exponent and certain of their appliccations / E.M. Nikishin // Matem. sbornik. – 1975. – 96(1). – P. 3–40 (in Russian).

[26] Šleževičienė, R. The joint universality for twists of Dirichlet series with multiplicative coefficients by characters / R. Šleževičienė // Analytic and Probab. Methods in Number Theory, Proc. the Third Palanga Conf., A. Dubickas et al (Eds), TEV 2002. – P. 309–319.

[27] Voronin, S.M. Theorem on the "universality" of the Riemann zeta-function / S.M. Voronin // Math. USSR Izv. – 1975. – 9(2). – P. 443–453.

[28] Voronin, S.M. On the functional independence of Dirichlet L-functions / S.M. Voronin // Acta Arith. – 1975. – 27. – P. 493-503 (in Russian).

[29] Wintner, A. On the asymptotic repartition of the values of real almost periodic functions / A. Wintner // Amer. J. Math. – 1932. – 54. – P. 334–345.

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РАСПРЕДЕЛЕНИЕ ЗНАЧЕНИЙ ОБОБЩЕННЫХ РЯДОВ ДИРИХЛЕ

 Дж. Жонас[2]

В работе дан обзор некоторых вероятностных результатов и их приложений к универсальности обобщенных рядов Дирихле.

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[2]Дженис Жонас (genys@splius.lt), кафедра математики и информатики университета Шауляя, ул. П. Висинского 19, LT-77156 Шауляй, Литва.