ON L-FUNCTION OF ELLIPTIC CURVES

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In the paper a survey on universality theorems for L-functions of elliptic curves over the field of rational numbers as well as their derivatives is presented.

Introduction

Elliptic curves are one of the most important objects in algebraic geometry and, in general, in mathematics. The theory of elliptic curves is rather complicated and wattled by many conjectures. On the other hand, the elliptic curves have many practical applications, for example, to cryptography, to factoring positive integers and primality testing. To study the properties of elliptic curves H.Hasse introduced L-functions attached to these curves.

Let E be an elliptic curve over the field of rational numbers $\mathbb{Q}$ defined by the Weierstrass equation

$$y^2 = x^3 + ax + b \quad a, b \in \mathbb{Z}.$$

We assume that the cubic x^3+ax+b has not a multiple root. Denote by $\Delta = -16(4a^3 + 27b^2)$ the discriminant of the curve E, and suppose that $\Delta \neq 0$. Then the roots of the cubic x^3+ax+b are distinct, and the curve E is non-singular.

For each prime p, denote by $\nu(p)$ the number of solutions of the congruence

$$y^2 \equiv x^3 + ax + b \ (mod p),$$

and let $\lambda(p) = p - \nu(p)$. Let $s = \sigma + it$ be a complex variable. Then the L-function of the elliptic curve E is the Euler product

$$L_E(s) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}.$$

In view of the Hasse estimate

$$|\lambda(p)| < 2\sqrt{p},$$

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the infinite product for $L_E(s)$ converges absolutely and uniformly on compact subsets of the half-plane $D_a = \left\{s \in \mathbb{C} : \sigma > \frac{3}{2}\right\}$, and defines there an analytic function with no zeros. The function $L_E(s)$ also can be written in the form of the Dirichlet series

$$L_E(s) = \sum_{m=1}^{\infty} \frac{\lambda(m)}{m^s},$$

where

$$\lambda(m) = \prod_{p^\alpha \| m} \lambda(p^\alpha),$$

and $p^\alpha \parallel m$ means that $p^\alpha \mid m$ but $p^{\alpha+1} \nmid m$, and the series also converges absolutely in D_a.

1. Analytic properties of the function $L_E(s)$

Analytic continuation of the function $L_E(s)$ and its universality is closely related to those of L–function of certain modular forms. Therefore, we start with some facts from the theory of modular forms.

Denote by $SL(2, \mathbb{Z})$ the full modular group, i.e.

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Furthermore, for a positive integer q, define

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \equiv 0 \ (\mathrm{mod}\ \ \mathrm{q}) \right\}.$$

Then $\Gamma_0(q)$ is a subgroup of $SL(2, \mathbb{Z})$, and it is called Hecke's or congruence subgroup mod q.

Now let $U = \{z \in \mathbb{C} : z = x + iy, \quad i = \sqrt{-1}, \quad y > 0\}$ be the upper half-plane together with ∞. The rational numbers and ∞ are called cusps. Let $F(s)$ be a holomorphic on U function, and suppose that, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, the functional equation

$$F\left(\frac{az + b}{cz + d}\right) = (cz + d)^\kappa F(z) \tag{1}$$

with some even positive integer κ is satisfied. Then

$$F(z) = \sum_{m=-\infty}^{\infty} c(m) \mathrm{e}^{2\pi i m z}$$

is the Fourier series expansion of $F(z)$ at infinity. The function $F(s)$ is called holomorphic at infinity if $c(m) = 0$ for $m < 0$, and vanishing at infinity if $c(m) = 0$ for $m \leqslant 0$. Moreover, $F(z)$ is called holomorphic and vanishing at other cusps if the function

$$(cz+d)^{-\kappa}F\left(\frac{az+b}{cz+d}\right)$$

is holomorphic and vanishing at infinity for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ respectively. If $F(z)$ is holomorphic at the cusps, then it is called a modular form of weight κ. In this case, the Fourier series expansion at infinity of $F(z)$ is

$$F(z) = \sum_{m=0}^{\infty} c(m)e^{2\pi imz}. \tag{2}$$

If the modular form $F(z)$ of weight κ vanishes at the cusps, then it is called a cusp form of weight κ, and

$$F(z) = \sum_{m=1}^{\infty} c(m)e^{2\pi imz}$$

is its Fourier series expansion at infinity. If equation (1) is satisfied for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$, then the cusp form $F(z)$ is called a cusp form of weight κ and level q.

The Ramanujan cusp form

$$\Delta(z) = e^{2\pi imz}\prod_{m=1}^{\infty}\left(1-e^{2\pi imz}\right)^{24} = \sum_{m=1}^{\infty}\tau(m)e^{2\pi imz}$$

is a classical example of cusp forms for $SL(2,\mathbb{Z})$. Its weight is 12, and $\tau(m)$ is called the Ramanujan function. The function $\tau(m)$ is multiplicative.

Denote by $S_\kappa(\Gamma_0(q))$ the space of all cusp forms of weight κ and level q. An element F of $S_\kappa(\Gamma_0(q))$ is called a Hecke's eigenform if F is an eigenfunction for all Hecke operators

$$(T(m)f)(z) = m^{\kappa-1}\sum_{\substack{0<d|m \\ ad=m}} d^{-\kappa} f\left(\frac{az+b}{d}\right).$$

If $q_1 \mid q$, then an element F of $S_\kappa(\Gamma_0(q_1))$ can be also an element of $S_\kappa(\Gamma_0(q))$. An element of $S_\kappa(\Gamma_0(q))$ is called a newform if it is a Hecke eigenform and if it is not a cusp form of level less than q. Let $F(s)$ be a cusp form of weight κ with the Fourier series expansion (2). Then the function

$$L(s,F) = \sum_{m=1}^{\infty}\frac{c(m)}{m^s}$$

is called the L-function of the cusp form $F(z)$. The series for $L(s,F)$ converges absolutely for $\sigma > \frac{\kappa+1}{2}$, moreover, $L(s,F)$ is analytically continuable to an entire function.

Now we will formulate the principal properties of the function $L_E(s)$. For long time, these properties were known as the conjectures.

Conjecture A (H. Hasse). The function $L_E(s)$ is analytically continuable to an entire function and satisfies the functional equation

$$\left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s)L_E(s) = \eta\left(\frac{\sqrt{q}}{2\pi}\right)^{2-s} \Gamma(2-s)L_E(2-s),$$

where q is a positive integer composed of prime factors of the discriminant Δ, $\eta = \pm 1$ is the root number, and $\Gamma(s)$, as usual, denotes the Euler gamma-function.

Conjecture B (Shimura–Taniyama–Weil). The Fourier series

$$F(z) = \sum_{m=1}^{\infty} \lambda(m) e^{2\pi i m z}$$

is a newform of weight 2 for some $\Gamma_0(q)$.

Now Conjectures A and B are proved. First they were proved by R. Taylor and A. Wiles [15] for semistable elliptic curves, and this succeeded the proof of the last Fermat problem. We recall that in the semistable case there is no additive reduction but only multiplicative one is.

Recently, Conjectures A and B were proved completely by C. Breuil, B. Conrad, F. Diamond and R. Taylor [2]. Therefore, analytic properties of the function $L_E(s)$ coincide with those of L-functions of newforms of weight 2.

2. Universality theorem of continuous type

The universality is a very interesting property of zeta and L-functions. J. Marcinkiewicz was the first who in 1935 used the name of the universality.

The first universality theorem for the Riemann zeta-function $\zeta(s)$ defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and by analytic continuation elsewhere, was discovered by S.M. Voronin in 1975 [13]. Let $0 < r < \frac{1}{4}$, and let $f(s)$ be a continuous non-vanishing function on the disc $|s| \leqslant r$ which is analytic in the interior of this disc. Then S. M. Voronin proved that, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s| \leqslant r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - f(s) \right| < \varepsilon.$$

Later, S.M. Gonek, A. Reich, B. Bagchi, A. Laurinčikas, K. Matsumoto, R. Garunkštis, J. Steuding, W. Schwarz, H. Mishou, R. Kačinskaitė, R. Šleževičienė, J. Ignatavičiūtė, J. Genys, H. Nagoshi and others generalized and improved the Voronin theorem. It turns out that a given analytic function $f(s)$ can be approximated by translations of $\zeta(s)$ uniformly on more general sets than a disc. Denote by meas$\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let, for $T > 0$,

$$\nu_T(...) = \frac{1}{T}\text{meas}\{\tau \in [0, T] : ...\},$$

where in place of dots a condition satisfied by τ is to be written. Then the last version of the Voronin theorem is contained in the following statement, see, for example, [11]. Let K be a compact subset of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complement. Let $f(s)$ be a continuous and non-vanishing on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right) > 0. \tag{3}$$

The latter theorem shows that there exist many translations $\zeta(s+i\tau)$ which approximate a given analytic function $f(s)$: the set of τ in (3) has a positive lower density.

The majority of classical zeta and L–functions are universal in the Voronin sense. The Linnik–Ibragimov conjecture says that all functions in some half-plane given by the Dirichlet series, analytically continuable to the left of the absolute convergence half-plane and satisfying some natural growth conditions are universal in the Voronin sense. All recent results on the universality of the Dirichlet series support that conjecture.

The aim of this paper is to give a survey on the universality of L–functions of elliptic curves. The universality of L–functions of newforms has been proved in [11]. From this the universality of $L_E(s)$ follows. Let $D = \{s \in \mathbb{C} : 1 < \sigma < \frac{3}{2}\}$.

Theorem 1. *Suppose that E is a non-singular elliptic curve over the field of rational numbers. Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K} |L_E(s + i\tau) - f(s)| < \varepsilon \right) > 0.$$

3. Generalizations of Theorem 1

Theorem 1 can be generalized in the two directions: for powers of $L_E(s)$ as well as a weighted universality theorem for $L_E(s)$ can be obtained.

Theorem 2. *Suppose that E is a non-singular elliptic curve over the field of rational numbers. Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$ and $k \in \mathbb{N}$,*

$$\liminf_{T\to\infty} \nu_T \left(\sup_{s\in K} |L_E^k(s + i\tau) - f(s)| < \varepsilon \right) > 0.$$

Proof of Theorem 2 is given in [7].

Now let T_0 be a fixed positive number, and let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$. Define

$$U = U(T, w) = \int_{T_0}^{T} w(t)\mathrm{d}t$$

and suppose that

$$\lim_{T\to\infty} U(T, w) = +\infty. \tag{4}$$

For the proof of Theorems 1 and 2, limit theorems in the sense of weak convergence of probability measures in the space of analytic functions for the considered functions are applied. On the other hand, the identification of the limit measures in these theorems is based on the ergodic theory, more precisely, on the Birkhoff—Khinchine theorem. A weighted analogue the latter theorem is not known. Therefore, to obtain a weighed universality theorem for the function $L_E(s)$ we need a certain additional condition.

Denote by $E\eta$ the expectation of the random variable η. Let $X(\tau, \omega)$ be an ergodic process, $E|X(\tau, \omega)| < \infty$, with sample paths almost surely integrable in the Riemann sense over every finite interval. Suppose that

$$\frac{1}{U}\int_{T_0}^{T} w(\tau)X(t+\tau, \omega)\mathrm{d}\tau = EX(0, \omega) + o\,(1+|t|)^{\delta} \tag{5}$$

almost surely for all $t \in \mathbb{R}$ with some $\delta > 0$ as $T \to \infty$. Let I_A denote the indicator function of the set A.

Theorem 3. *Let the function $w(t)$ satisfy conditions (4) and (5), and let K and $f(s)$ be the same as in Theorem 1. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau:\sup\limits_{s\in K}|L_E(s+i\tau)-f(s)|<\varepsilon\}}\mathrm{d}\tau > 0.$$

Proof of Theorem 3 is given in [3].

The universality of the derivative $L'_E(s)$ is contained in the following statement.

Theorem 4 [8]. *Let K be a compact subset of the strip D with connected complement, and let $f(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}|L'_E(s+i\tau) - f(s)| < \varepsilon\right) > 0.$$

Note that, differently from Theorems 4, the function $f(s)$ can be vanishing on K.

The universality of functions implies their functional independence. Note that the problem of independence of functions comes back to D. Hilbert. S.M. Voronin [14] obtained the functional independence of $\zeta(s)$. Let F_l, $l = 0, 1, ..., n$, be continuous functions, and let the equality

$$\sum_{l=0}^{n} s^l F_l\left(\zeta(s), \zeta'(s), ..., \zeta^{(N-1)}(s)\right) = 0$$

be valid identically for s. Then $F_l \equiv 0$ for $l = 0, 1, ..., n$.

Functions $L_E^k(s)$ are functional independent too.

Theorem 5. *Let $h_0, ..., h_M$ be continuous functions on $\mathbb{C}^n$. If*

$$\sum_{m=0}^{M} s^m h_m\left(L_E^k(s), kL_E^{k-1}(s)L_E'(s), ..., (L_E(s))^{(n-1)}\right) \equiv 0,$$

then $h_m \equiv 0$ for $m = 0, 1, ..., M$.

Proof of Theorem 5 is given in [7].

4. Discrete universality

All stated above universality theorems are of continuous type: in them translations the imaginary part of the complex variable varies continuously in the interval $[0, T]$. Also, a discrete version of universality theorems exists. In this case, the imaginary part of the complex variable in translations takes values from some arithmetical progression. Let, for $N \in \mathbb{N}$,

$$\mu_N(...) = \frac{1}{N+1} \sharp\{0 \leqslant m \leqslant N : ...\},$$

where in place of dots a condition satisfied by m is to be written, and let $h > 0$ be a fixed number.

The discrete universality of the Riemann zeta-function has been considered by S.M. Voronin [14] and B. Bagchi [1]. R. Kačinskaitė obtained [10] the discrete universality for the Matsumoto zeta-function [12] with special h and under some additional conditions. J. Ignatavičiūtė proved [9] a discrete universality theorem for the Lerch zeta-function

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s}, \quad \sigma > 1,$$

with transcendental parameter α, and h such that $\exp\left\{\frac{2\pi}{h}\right\}$ is rational. In general, results on the discrete universality for functions given by the Dirichlet series are not numerous. For the function $L_E(s)$ the following statement is valid [6].

Theorem 6. *Suppose that* $\exp\left\{\frac{2\pi k}{h}\right\}$ *is an irrational number for all* $k \in \in \mathbb{Z} \setminus \{0\}$. *Let* K *and* $f(s)$ *be the same as in Theorem 1. Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \mu_N\left(\sup_{s\in K} |L_E(s+imh) - f(s)| < \varepsilon\right) > 0.$$

Theorem 6 shows that the set $\{m : m = 0, 1, ...\}$ such that $L_E(s + imh)$ approximates a given analytic function is sufficiently rich: it has a positive lower density. Since by the Hermite–Lindemann theorem e^a with an algebraic number $a \neq 0$, is irrational, Theorem 6 holds, for example, with $h = 2\pi$. On the other hand, Theorem 6 as well as continuous universality Theorems 1–4 are non-effective in the sense that there are not known at least one m or τ having approximation properties.

The case of h when $\exp\{\frac{2\pi k}{h}\}$ is rational for some $k \neq 0$ is more complicated than that of Theorem 6. It turns out that the assertion of Theorem 6 remains valid also in this case, however its proof has some essential differences.

The proof of Theorem 6 is based on a limit theorem in the space $H(D)$ of analytic on D functions for the function $L_E(s)$. Denote by γ the unit circle on the complex plane $\mathbb{C}$, and define the infinite-dimensional torus

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime p. With the product topology and pointwise multiplication Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(S)$ denotes the class of Borel sets of the space S, the probability Haar measure m_H can be defined. This gives a probability space $(\Omega, \mathcal{B}(\Omega), m_{hH})$. Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_p. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define an $H(D)$-valued random element $L_E(s, \omega)$ by

$$L_E(s,\omega) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1},$$

and denote be P_{L_E} its distribution. Then the proof of Theorem 6 uses the following probabilistic limit theorem. The probability measure

$$\mu_N(L_E(s + imh) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the P_{L_E} as $N \to \infty$.

If $\exp\left\{\frac{2\pi k}{h}\right\}$ is rational for some integers $k \neq 0$, then a modification of the latter theorem is used. Let

$$k_0 = \min\left\{k \in \mathbb{N} : \exp\left\{\frac{2\pi k}{h}\right\} \text{ is rational}\right\}.$$

If $\exp\left\{\frac{2\pi k}{h}\right\}$ is rational, then k is a multiple of k_0. Denote

$$\exp\left\{\frac{2\pi k_0}{h}\right\} = \frac{m_0}{n_0}, \quad m_0, n_0 \in \mathbb{N}, (m_0, n_0) = 1,$$

and define

$$\Omega_h = \{\omega \in \Omega : \omega(m_0) = \omega(n_0)\},$$

where

$$\omega(m) = \prod_{p^\alpha \| m} \omega^\alpha(p).$$

Ω_h is a closed subgroup of Ω, therefore it is also a compact topological group. Thus we obtain a probability space $(\Omega_h, \mathcal{B}(\Omega_h), m_{hH})$, where m_{hH} is the probability Haar measure on $(\Omega_h, \mathcal{B}(\Omega_h))$. Now on the latter probability space define an $H(D)$-valued random element $L_E(s, \omega_h)$ by

$$L_E(s, \omega_h) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)\omega_h(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)\omega_h(p)}{p^s} + \frac{\omega_h^2(p)}{p^{2s-1}}\right)^{-1}.$$

Then it is proved [5] that if $\exp\left\{\frac{2\pi k}{h}\right\}$ is rational for some $k \neq 0$, then the probability measure

$$\mu_N\left(L_E(s + imh) \in A\right), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the distribution of the random element $L_E(s, \omega_h)$ as $N \to \infty$. From this theorem an analogue of Theorem 6 follows.

Theorem 7. *Suppose that* $\exp\left\{\frac{2\pi k}{h}\right\}$ *is rational for some* $k \neq 0$. *Then the assertion of Theorem 6 is true.*

Proof of Theorem 7 is given in [5].

5. A weighted discrete universality theorem

We present a very simple discrete universality theorem with weight for the function $L_E(s)$. Let $w(x)$ be a non-negative function on $[0, \infty)$. Suppose that

$$U = U(N, w) = \sum_{m=0}^{N} w(m) \to \infty$$

as $N \to \infty$. We prove the following statement. Let, for brevity,

$$v_N(\ldots) = \frac{1}{U}\sum_{\substack{m=0 \\ \ldots}}^{N} w(m),$$

where in place of dots a condition satisfied by m is to be written.

Theorem 8. *Suppose that* $w(x)$ *is a continuous non-increasing function on* $[0,\infty)$*, and that* $\exp\left\{\frac{2\pi k}{h}\right\}$ *is an irrational number for all* $k \in \mathbb{Z}\setminus\{0\}$*. Let* K *be a compact subset of the strip* $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ *with connected complement. Let* $f(s)$ *be a continuous and non-vanishing on* K *function which is analytic in the interior of* K*. Then, for every* $\varepsilon > 0$*,*

$$\liminf_{N\to\infty} \nu_N \left(\sup_{s\in K} |L_E(s+imh) - f(s)| < \varepsilon\right) > 0.$$

The proof of Theorem 8 [4] is based on a discrete limit theorem, in this case with weight, in the sense of weak convergence of probability measures in the space of analytic functions.

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L-ФУНКЦИИ НА ЭЛЛИПТИЧЕСКИХ КРИВЫХ

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В работе дан обзор теорем универсальности для *L*-функций эллиптических кривых над полем рациональных чисел, а также для их производных.

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