МЕХАНИКА

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ON DETERMINATION OF LINEAR FREQUENCIES OF BENDING VIBRATIONS OF PIEZOELECTRIC SHELLS AND PLATES BY EXACT AND AVERAGED TREATMENT

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In this paper the derivation and numerical solution of disspersion relations for frequencies of free bending vibrations for piezoelectric cylindrical thin shells with longitudinal polarization and plates with normal polarization is made. Solution is done by exact space treatment and by Kirchhoff hypothesis. Comparision of obtained tables shows that frequencies by exact and based on Kirchhoff hypothesis are quite different.

Introduction

The bending vibrations of magnetoelastic shells and plates by averaged treatment based on classic theory are considered in [1–5]. By the new space treatment at first developed for elastic plates in [6], the magnetoelstic vibrations of plates and shells are considered in [7–9]. The dispersion relation for Lamb waves in piezoelastic strip by exact treatment is obtained in [10], [11], where are obtained numerically five modes of mentioned waves, but is not made carefully investigation of solution of transcendent dispersion equation corresponding to law of relation of frequency from wave number for bending waves for thin plates. The above mentioned investigation is carried out analytically in [7–9] for magnetoelastic plates and it is shown that almost for all cases the results obtained by exact solution are distinguished essentially from averaged solution based on Kirchhoff hypothesis, which formerly give excellent results for elastic plates [6]. In present paper by space treatment of [6–11] are determined analytically and numerically the frequencies of free bending vibrations of the piezoelectric cylindrical shell with longitudinal polarisation and the comparison with averaged treatment is carried out. Besides the same investigation for piezoelectric plate with transverse polarization is carried out and are made and

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compared calculations of frequencies by space and averaged treatment. As for shell as for plate it is shown that Kirchhoff hypotesis for determination of free bending vibrations frequencies for piezoelectric is not applicable.

1. Statement of problem and solution for cylindrical shell

Let us consider the infinite cylindrical shell of small thickness $2h$ and radius of middle section R made from piezoelectric with elastic constants $C_{11}, C_{12}, C_{13}, C_{44}$ and piezomodulus e_{31}, e_{33}, e_{15} [10]. For the case of axial polarization of shell choosing coordinate along axis of cylinder and as radial coordinate one can write the stresses and electrical induction components in shell [10] as

$$\begin{aligned}
\sigma_{rr} &= C_{11}\frac{\partial u_r}{\partial r} + C_{12}\frac{u_r}{r} + C_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial \phi}{\partial z},\\
\sigma_{zz} &= C_{13}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + C_{33}\frac{\partial u_z}{\partial z} + e_{33}\frac{\partial \phi}{\partial z},\\
\sigma_{rz} &= C_{44}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) + e_{15}\frac{\partial \phi}{\partial r},\\
\sigma_{\vartheta\vartheta} &= C_{12}\frac{\partial u_r}{\partial r} + C_{11}\frac{u_r}{r} + C_{13}\frac{\partial u_z}{\partial z} + e_{31}\frac{\partial \phi}{\partial z},\\
D_r &= e_{15}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) - \varepsilon_{11}\frac{\partial \phi}{\partial r},\\
D_z &= e_{31}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r}\right) + e_{33}\frac{\partial u_z}{\partial z} - \varepsilon_{33}\frac{\partial \phi}{\partial z},
\end{aligned} \tag{1.1}$$

where u_r, u_z are displacements components, ϕ– potential of electrical field, $\varepsilon_{11}, \varepsilon_{33}$ are dielectric permeabilities.

Then equations of motion and induction yield [10]

$$\begin{aligned}
&C_{11}\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2}\right) + C_{44}\frac{\partial^2 u_r}{\partial z^2} + (C_{13}+C_{44})\frac{\partial^2 u_z}{\partial r \partial z} +\\
&\qquad + (e_{31}+e_{15})\frac{\partial^2 \phi}{\partial r \partial z} = \rho\frac{\partial^2 u_r}{\partial t^2},
\end{aligned} \tag{1.2}$$

$$\begin{aligned}
&C_{44}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) + C_{33}\frac{\partial^2 u_z}{\partial z^2} + (C_{13}+C_{44})\left(\frac{\partial^2 u_r}{\partial r \partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z}\right) +\\
&\qquad + e_{15}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right) + e_{33}\frac{\partial^2 \phi}{\partial z^2} = \rho\frac{\partial^2 u_z}{\partial t^2},\\
&e_{15}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) + e_{33}\frac{\partial^2 u_z}{\partial z^2} + (e_{15}+e_{31})\left(\frac{\partial^2 u_r}{\partial r \partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z}\right) -\\
&\qquad - \varepsilon_{11}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right) - \varepsilon_{33}\frac{\partial^2 \phi}{\partial z^2} = 0,
\end{aligned} \tag{1.3}$$

where is density. The exact particular solution of (1.2), (1.3) for propagated along axis plane wave as in corresponding piezoelectric plate [11] and in magnetoelastic shell and plate [7–9] can be looked for in space treatment in form

$$\begin{aligned}
&\xi_j = r\nu_j, \;\; j = 1,2,3,\\
&u_r = A_j I_1\left(\xi_j\right) e^{ikz-i\omega t} + A'_j K_1\left(\xi_j\right) e^{ikz-i\omega t} + c.c.,\\
&u_z = B_j I_0\left(\xi_j\right) e^{ikz-i\omega t} + B'_j K_0\left(\xi_j\right) e^{ikz-i\omega t} + c.c.,\\
&\phi = \phi_j I_0\left(\xi_j\right) e^{ikz-i\omega t} + \phi'_j K_0\left(\xi_j\right) e^{ikz-i\omega t} + c.c.,
\end{aligned} \tag{1.4}$$

where $I_{0,1}$, $K_{0,1}$ are Bessel functions of imagine argument on is carried out summation from 1 to 3. On account relations

$$\begin{aligned}
&I'_0(\xi) = I_1(\xi), \; K'_0(\xi) = -K_1(\xi),\\
&\frac{dI_1(\xi)}{d\xi} + \frac{1}{\xi} I_1(\xi) = I_0(\xi), \; \frac{dK_1(\xi)}{d\xi} + \frac{1}{\xi} K_1(\xi) = -K_0(\xi)
\end{aligned} \tag{1.5}$$

one can from (1.2)–(1.5) obtain

$$\begin{aligned}
&-iA_j\left(C_{11}\nu_j^2 - C_{44}k^2 + \rho\omega^2\right) + \left(C_{13} + C_{44}\right)\nu_j k B_j +\\
&+\left(e_{31} + e_{15}\right)\nu_j k \phi_j = 0,\\
&A_j \nu_j ik\left(C_{13} + C_{44}\right) + \left(C_{44}\nu_j^2 - C_{33}k^2 + \rho\omega^2\right) B_j +\\
&\phi_j\left(e_{15}\nu_j^2 - e_{33}k^2\right) = 0,\\
&\left(e_{31} + e_{15}\right) A_j \nu_j ik + B_j\left(e_{15}\nu_j^2 - e_{33}k^2\right) + \phi_j\left(-\varepsilon_{11}\nu_j^2 + \varepsilon_{33}k^2\right) = 0.
\end{aligned} \tag{1.6}$$

The equation for ν_j^2 is distinguished from equation of [11] for plate with normal polarization and yields

$$\begin{aligned}
&\nu_j = \lambda_j k, \; \frac{\rho\omega^2}{C_{11}k^2} = \nu^2, \; \frac{C_{44}}{C_{11}} = \mu_1, \; \frac{C_{13} + C_{44}}{C_{11}} = \mu_2, \; \frac{C_{33}}{C_{11}} = \mu_4,\\
&\frac{C_{12}}{C_{11}} = \frac{1}{2},\\
&\mu_5 = \frac{e_{33}}{e_{31} + e_{15}}, \; \mu_6 = \frac{e_{15}}{e_{31} + e_{15}}, \; \mu_7^2 = \frac{\varepsilon_{11}}{\varepsilon_{33}}, \; k_1^2 = \frac{\left(e_{31} + e_{15}\right)^2}{C_{11}\varepsilon_{33}},\\
&k_2 = \mu_6 k_1^2, \; k_3 = \mu_5 k_1^2, \; \det\left\|a_{ij}\right\| = 0,
\end{aligned} \tag{1.7}$$

where

$$\begin{aligned}
&a_{11} = -\lambda_j^2 + \mu_1 - \nu^2, \; a_{12} = a_{21} = \mu_2\lambda_j, \; a_{13} = k_1^2\lambda_j,\\
&a_{22} = \mu_1\lambda_j^2 - \mu_4 + \nu^2,\\
&a_{23} = k_1^2\left(\mu_6\lambda_j^2 - \mu_5\right), \; a_{31} = \lambda_j, \; a_{32} = \mu_6\lambda_j^2 - \mu_5, \; a_{33} = 1 - \mu_7^2\lambda_j^2.
\end{aligned} \tag{1.8}$$

For elastic case when $\mu_5 = \mu_6 = k_1^2 = k_2 = k_3 = 0$ (1.7), (1.8) have two roots $\lambda_{2,3}^0$, and for piezoelectric one must seek solution of (1.7) starting from values of $\lambda_2 = \lambda_2^0$, $\lambda_3 = \lambda_3^0$, $\lambda_1 = \dfrac{1}{\mu_7}$. Denoting $\dfrac{(e_{31}+e_{15})\,\phi_j}{C_{11}} = \varphi_j$ one can obtain from (1.6)

$$\begin{aligned} &iA_j a_{11} + B_j a_{12} + \varphi_j a_{13} = 0,\\ &iA_j a_{21} + B_j a_{22} + \varphi_j a_{23} = 0,\\ &iA_j a_{31} + B_j a_{32} + \varphi_j a_{33} = 0 \end{aligned} \tag{1.9}$$

and [11]

$$iA_j = \alpha_j U_j,\ B_j = \beta_j U_j,\ \varphi_j = \gamma_j U_j, \tag{1.10}$$

where

$$\begin{aligned} &\alpha_j\left(\lambda_j\right) = a_{12}a_{23} - a_{13}a_{22},\ \beta_j\left(\lambda_j\right) = a_{21}a_{13} - a_{11}a_{23},\\ &\gamma_j\left(\lambda_j\right) = a_{11}a_{22} - a_{12}^2. \end{aligned} \tag{1.11}$$

For A'_j, $-B'_j$, $-\varphi'_j$ one obtains the same (1.11) equation expressed by U'_j. Then (1.4) gives

$$\begin{aligned} &u_r = -i\alpha_j U_j I_1\left(\xi_j\right)e^{ikz-i\omega t} - i\alpha_j U'_j K_1\left(\xi_j\right)e^{ikz-i\omega t} + c.c.,\\ &u_z = \beta_j U_j I_0\left(\xi_j\right)e^{ikz-i\omega t} - \beta_j U'_j K_0\left(\xi_j\right)e^{ikz-i\omega t} + c.c.,\\ &\frac{(e_{31}+e_{15})\,\phi}{C_{11}} = \gamma_j U_j I_0\left(\xi_j\right)e^{ikz-i\omega t} - \gamma_j U'_j K_0\left(\xi_j\right)e^{ikz-i\omega t} + c.c.,\\ &\frac{(e_{31}+e_{15})}{C_{11}}\phi = \varphi, \end{aligned} \tag{1.12}$$

where is carried out summation on from 1 to 3. Using (1.1), (1.5) and (1.12) one obtains

$$\begin{aligned} &\frac{\sigma_{rr}}{ikC_{11}} = -\alpha_j U_j \lambda_j I'_1\left(\xi_j\right)e^{ikz-i\omega t} - \frac{C_{12}}{C_{11}}\alpha_j U_j \lambda_j \frac{I_1\left(\xi_j\right)}{\xi_j}e^{ikz-i\omega t} +\\ &+\frac{C_{13}}{C_{11}}\beta_j U_j I_0\left(\xi_j\right)e^{ikz-i\omega t} - \alpha_j U'_j \lambda_j K'_1\left(\xi_j\right)e^{ikz-i\omega t} -\\ &-\frac{C_{12}}{C_{11}}\alpha_j U'_j \lambda_j \frac{K_1\left(\xi_j\right)}{\xi_j}e^{ikz-i\omega t} - \frac{C_{13}}{C_{11}}\beta_j U'_j K_0\left(\xi_j\right)e^{ikz-i\omega t} +\\ &+\frac{e_{31}}{e_{31}+e_{15}}U_j\gamma_j I_0\left(\xi_j\right)e^{ikz-i\omega t} - \frac{e_{31}}{e_{31}+e_{15}}U'_j\gamma_j K_0\left(\xi_j\right)e^{ikz-i\omega t} + c.c.,\\ &\frac{\sigma_{rz}}{kC_{44}} = n_j^* U_j I_1\left(\xi_j\right)e^{ikz-i\omega t} + n_j^* U'_j K_1\left(\xi_j\right)e^{ikz-i\omega t} + c.c.,\\ &n_j^* = \alpha_j + \beta_j\lambda_j + \frac{\mu_6 C_{11}}{C_{44}}\lambda_j\gamma_j,\\ &\frac{D_r}{k\left(e_{31}+e_{15}\right)} = t_j^* U_j I_1\left(\xi_j\right)e^{ikz-i\omega t} + t_j^* U'_j K_1\left(\xi_j\right)e^{ikz-i\omega t} + c.c.,\\ &\qquad t_j^* = \mu_6\alpha_j + \mu_6\beta_j\lambda_j - \frac{\varepsilon_{11}C_{11}}{\left(e_{31}+e_{15}\right)^2}\gamma_j\lambda_j. \end{aligned} \tag{1.13}$$

The boundary conditions on shell surfaces as in case of piezoelectric plates give [11]

$$\begin{gathered}
\sigma_{rr}(R \pm h, z) = 0, \ \sigma_{rz}(R \pm h, z) = 0, \\
\varphi(R \pm h, z) = \bar{\varphi}(R \pm h, z), \ D_r(R \pm h, z) = \bar{D}_r(R \pm h, z)\frac{C_{11}}{e_{31} + e_{15}}.
\end{gathered} \tag{1.14}$$

The dimensionless potential $\bar{\varphi}(r, z, t)$ in dielectric out of shell satisfy the equation $\frac{\partial^2 \bar{\varphi}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\varphi}}{\partial r} + \frac{\partial^2 \bar{\varphi}}{\partial z^2} = 0$ and one can look for solution outside of shell in form

$$\begin{aligned}
\bar{\varphi} &= \bar{\phi}_+ e^{ikz - i\omega t} K_0(kr) + c.c., \quad r > R + h, \\
\bar{\varphi} &= \bar{\phi}_- e^{ikz - i\omega t} I_0(kr) + c.c., \quad r < R - h.
\end{aligned} \tag{1.15}$$

For $\bar{D}_r = \varepsilon \frac{\partial \bar{\varphi}}{\partial r}$ one obtains

$$\begin{aligned}
\frac{1}{k}\bar{D}_r &= -\bar{\phi}_+ \varepsilon e^{ikz - i\omega t} K_1(kr) + c.c., \quad r > R + h, \\
\frac{1}{k}\bar{D}_r &= \bar{\phi}_- \varepsilon e^{ikz - i\omega t} I_1(kr) + c.c., \quad r < R - h.
\end{aligned} \tag{1.16}$$

The first line equations (1.14) give four equations

$$\begin{gathered}
\alpha_j U_j \lambda_j I_1'\left(\xi_j^\pm\right) + \frac{C_{12}}{C_{11}}\alpha_j U_j \lambda_j \frac{I_1\left(\xi_j^\pm\right)}{\xi_j^\pm} - \frac{C_{13}}{C_{11}}\beta_j U_j I_0\left(\xi_j^\pm\right) + \\
+\alpha_j U_j' \lambda_j K_1'\left(\xi_j^\pm\right) + \frac{C_{12}}{C_{11}}\alpha_j U_j \lambda_j \frac{K_1\left(\xi_j^\pm\right)}{\xi_j^\pm} + \frac{C_{13}}{C_{11}}\beta_j U_j' K_0\left(\xi_j^\pm\right) - \\
-(1 - \mu_6) U_j \gamma_j I_0\left(\xi_j^\pm\right) + (1 - \mu_6) U_j' \gamma_j K_0\left(\xi_j^\pm\right) = 0, \\
n_j^* U_j I_1\left(\xi_j^\pm\right) + n_j^* U_j' K_1\left(\xi_j^\pm\right) = 0, \ \xi_j^\pm = (R \pm h) k \lambda_j,
\end{gathered} \tag{1.17}$$

where is carried out summation on from 1 to 3. The last conditions in (1.14) and (1.12), (1.15) yield.

$$\begin{gathered}
\gamma_j U_j I_0\left(\xi_j^+\right) - \gamma_j U_j' K_0\left(\xi_j^+\right) = \bar{\phi}_+ K_0\{(R + h)k\}, \\
\gamma_j U_j I_0\left(\xi_j^-\right) - \gamma_j U_j' K_0\left(\xi_j^-\right) = \bar{\phi}_- I_0\{(R - h)k\}, \\
t_j^* U_j I_1\left(\xi_j^+\right) + t_j^* U_j' K_1\left(\xi_j^+\right) = \frac{C_{11}}{(e_{31} + e_{15})^2}\bar{\phi}_+ K_1\{(R + h)k\}, \\
t_j^* U_j I_1\left(\xi_j^-\right) + t_j^* U_j' K_1\left(\xi_j^-\right) = -\frac{C_{11}}{(e_{31} + e_{15})^2}\bar{\phi}_- I_1\{(R - h)k\},
\end{gathered}$$

or excluding of $\overline{\phi}_-$, $\overline{\phi}_+$

$$\begin{aligned}
&\gamma_j U_j I_0\left(\xi_j^+\right) - \gamma_j U_j' K_0\left(\xi_j^+\right) = \\
&= -\frac{(e_{31}+e_{15})^2}{\varepsilon C_{11}}\left\{t_j^* U_j I_1\left(\xi_j^+\right) + t_j^* U_j' K_1\left(\xi_j^+\right)\right\}\frac{K_0\{(R+h)k\}}{K_1\{(R+h)k\}}, \\
&\gamma_j U_j I_0\left(\xi_j^-\right) - \gamma_j U_j' K_0\left(\xi_j^-\right) = \\
&= -\frac{(e_{31}+e_{15})^2}{\varepsilon C_{11}}\left\{t_j^* U_j I_1\left(\xi_j^-\right) + t_j^* U_j' K_1\left(\xi_j^-\right)\right\}\frac{I_0\{(R-h)k\}}{I_1\{(R-h)k\}},
\end{aligned} \tag{1.18}$$

where is carried out summation on from 1 to 3. Equations (1.17), (1.18) relate all U_j, U_j' by homogeneous linear system, where determinant equation is

$$\begin{vmatrix}
\Pi_1^+ & \Pi_2^+ & \Pi_3^+ & \mathrm{M}_1^+ & \mathrm{M}_2^+ & \mathrm{M}_3^+ \\
\Pi_1^- & \Pi_2^- & \Pi_3^- & \mathrm{M}_1^- & \mathrm{M}_2^- & \mathrm{M}_3^- \\
\mathrm{P}_1^+ & \mathrm{P}_2^+ & \mathrm{P}_3^+ & \Omega_1^+ & \Omega_2^+ & \Omega_3^+ \\
\mathrm{P}_1^- & \mathrm{P}_2^- & \mathrm{P}_3^- & \Omega_1^- & \Omega_2^- & \Omega_3^- \\
\mathrm{N}_1^+ & \mathrm{N}_2^+ & \mathrm{N}_3^+ & \Lambda_1^+ & \Lambda_2^+ & \Lambda_3^+ \\
\mathrm{N}_1^- & \mathrm{N}_2^- & \mathrm{N}_3^- & \Lambda_1^- & \Lambda_2^- & \Lambda_3^-
\end{vmatrix} = 0, \tag{1.19}$$

$$\begin{aligned}
&\Pi_j^\pm = \alpha_j\lambda_j I_1'\left(\xi_j^\pm\right) + \frac{C_{12}}{C_{11}}\alpha_j\lambda_j\frac{I_1\left(\xi_j^\pm\right)}{\xi_j^\pm} - \frac{C_{13}}{C_{11}}\beta_j I_0\left(\xi_j^\pm\right) - \\
&-(1-\mu_6)\gamma_j I_0\left(\xi_j^\pm\right), \\
&\mathrm{M}_j^\pm = \alpha_j\lambda_j K_1'\left(\xi_j^\pm\right) + \frac{C_{12}}{C_{11}}\alpha_j\lambda_j\frac{K_1\left(\xi_j^\pm\right)}{\xi_j^\pm} + \frac{C_{13}}{C_{11}}\beta_j K_0\left(\xi_j^\pm\right) + \\
&+(1-\mu_6)\gamma_j K_0\left(\xi_j^\pm\right),
\end{aligned} \tag{1.20}$$

$$\mathrm{P}_j^\pm = n_j^* I_1\left(\xi_j^\pm\right),\ \Omega_j^\pm = n_j^* K_1\left(\xi_j^\pm\right),$$

$$\mathrm{N}_j^+ = \gamma_j I_0\left(\xi_j^+\right) - \frac{(e_{31}+e_{15})^2}{\varepsilon C_{11}}\frac{t_j^* I_1\left(\xi_j^+\right)}{K_1\{k(R+h)\}}K_0\{k(R+h)\},$$

$$\mathrm{N}_j^- = \gamma_j I_0\left(\xi_j^-\right) + \frac{(e_{31}+e_{15})^2}{\varepsilon C_{11}}\frac{t_j^* I_1\left(\xi_j^-\right)}{I_1\{k(R-h)\}}I_0\{k(R-h)\},$$

$$\Lambda_j^+ = -\gamma_j K_0\left(\xi_j^+\right) - \frac{(e_{31}+e_{15})^2}{\varepsilon C_{11}}\frac{t_j^* K_1\left(\xi_j^+\right)}{K_1\{k(R+h)\}}K_0\{k(R+h)\},$$

$$\Lambda_j^- = -\gamma_j K_0\left(\xi_j^-\right) + \frac{(e_{31}+e_{15})^2}{\varepsilon C_{11}}\frac{t_j^* K_1\left(\xi_j^-\right)}{I_1\{k(R-h)\}}I_0\{k(R-h)\}.$$

Where there is not summation by j.
We must carry out calculations for piezoelectric case (1.8), (1.19). For all values

of constants for BaTio3 are as follows

$$\mu_1 = \frac{1}{3},\ \mu_2 = \frac{5}{6},\ \mu_4 = 1,$$
$$\frac{C_{12}}{C_{11}} = \frac{1}{2},\ \mu_5 = 2,\ \mu_6 = \frac{3}{2},\ \mu_7 = 1,\ \frac{\varepsilon_{33}}{\varepsilon} = 10, \tag{1.21}$$
$$k_1^2 = \frac{n}{300},\ n = 4,\ 50,\ 100.$$

Placing $\lambda_{1,2,3}(\nu)$ from (1.7) in (2.4), (1.19), one must solve dispersion equation for small values of kh , i.e. for $h = 0.1$ cm, $k = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5$ 1/cm, $R = 10^3$ cm and obtain tables of $\nu = \nu(k)$ or $\omega = k\sqrt{\frac{C_{11}}{\rho}}\nu(k)$. Results are brought in table 1.

Table 1

$h = 0.1,\ R = 10^3$

$k_1^2 = \frac{n}{300}$	k				
	0.1	0.2	0.3	0.4	0.5
n=4	0.0145	0.0102	0.0083	0.0072	0.0064
n=50	0.0299	0.02118	0.0173	0.0149	0.0134
n=100	0.0366	0.0259	0.0211	0.0183	0.0164

2. The case of elastic cylindrical shell

For elastic shell one must put

$$e_{31} = 0,\ e_{33} = 0,\ e_{15} = 0,\ \varphi = 0. \tag{2.1}$$

and take place (1.4) for $u_r,\ u_z$. (1.7) yields

$$a_{11}a_{22} - a_{12}^2 = 0, \tag{2.2}$$

where a_{ik} are done by (1.8) and there are two roots $\lambda_{1,2}^0$. The relations (1.11) yield

$$\alpha_j(\lambda_j) = -a_{13}a_{22},\ \beta_j(\lambda_j) = -a_{12}a_{13},\ \ j = 1,2. \tag{2.3}$$

Then one has equations (1.17) on boundary of shell in which γ^j and n_j^* are given by (1.13).
In (1.17) unknown functions are . The determinant equation will give as in (1.19) for first four lines the same form without third and sixth columns

$$\begin{vmatrix} \Pi_1^+ & \Pi_2^+ & M_1^+ & M_2^+ \\ \Pi_1^- & \Pi_2^- & M_1^- & M_2^- \\ P_1^+ & P_2^+ & \Omega_1^+ & \Omega_2^+ \\ P_1^- & P_2^- & \Omega_1^- & \Omega_2^- \end{vmatrix} = 0, \tag{2.4}$$

where $\Pi_{1,2}^{\pm},\ M_{1,2}^{\pm},\ P_{1,2}^{\pm},\ \Omega_{1,2}^{\pm}$ are done in (1.20), where $\mu_5 = 0,\ \mu_6 = 0$.

3. Solution for cylindrical shell based on Kirchhoff hypothesis

For comparison with results of Kirchhoff hypothesis for piezoelectric shells one can assume that

$$\sigma_{rz} \approx 0,\ \ \sigma_{rr} \approx 0,\ \ u_r = A\sin kz,\ \ \varphi = \phi_0(r)\cos kz. \tag{3.1}$$

where multiplier $e^{-i\omega t}$ is omitted.
Then one obtains using (1.1)

$$\begin{aligned} &\frac{\partial u_z}{\partial r} = -\frac{\partial u_r}{\partial z} - \frac{e_{15}}{C_{44}}\frac{\partial\varphi}{\partial r}, \\ &u_z = (R-r)\frac{\partial u_r}{\partial z} - \frac{e_{15}}{C_{44}}\varphi + u(z), \\ &\frac{\partial u_r}{\partial r} = -\frac{C_{12}}{C_{11}}\frac{u_r}{r} - \frac{C_{13}}{C_{11}}\frac{\partial u_z}{\partial z} - \frac{e_{31}}{C_{11}}\frac{\partial\varphi}{\partial z}. \end{aligned} \tag{3.2}$$

Equations of motion are

$$\begin{aligned} &\frac{\partial\sigma_{rr}}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\vartheta\vartheta}}{r} = \rho\frac{\partial^2 u_r}{\partial t^2}, \\ &\frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = \rho\frac{\partial^2 u_z}{\partial t^2}, \end{aligned} \tag{3.3}$$

From (1.1), (3.2) one obtains

$$\begin{aligned} &\sigma_{zz} = \frac{u_r}{r}\left(C_{13} - \frac{C_{12}C_{13}}{C_{11}}\right) + \frac{\partial u_z}{\partial z}\left(C_{33} - \frac{C_{13}^2}{C_{11}}\right) + \\ &+\frac{\partial\varphi}{\partial z}\left(e_{33} - \frac{e_{31}C_{13}}{C_{11}}\right), \\ &\sigma_{\vartheta\vartheta} = \frac{u_r}{r}\left(C_{11} - \frac{C_{12}^2}{C_{11}}\right) + \frac{\partial u_z}{\partial z}\left(C_{13} - \frac{C_{12}C_{13}}{C_{11}}\right) + \\ &+\frac{\partial\varphi}{\partial z}\left(e_{31} - \frac{e_{33}C_{12}}{C_{11}}\right). \end{aligned} \tag{3.4}$$

Integrating (3.3) on r from $R-h$ to $R+h$, using that on $r = R\pm h,\ \sigma_{rr} = 0,\ \sigma_{rz} = 0$,and multiplying of second equation (3.3) by $R-h$ and integrating, one obtains equations

$$\begin{aligned} &\frac{\partial Q}{\partial z} - \frac{1}{R}\int\limits_{R-h}^{R+h}\sigma_{\vartheta\vartheta}dr = \rho\frac{\partial^2 u_r}{\partial t^2}2h,\ \ Q = \int\limits_{R-h}^{R+h}\sigma_{rz}dr, \\ &\frac{\partial\int_{R-h}^{R+h}\sigma_{zz}dr}{\partial z} = 0,\ \ \frac{\partial M}{\partial z} = Q,\ \ M = \int\limits_{R-h}^{R+h}(r-R)\,\sigma_{zz}dr, \end{aligned} \tag{3.5}$$

where small terms $\frac{\partial^2 u_z}{\partial t^2}$ as well as $\frac{Q}{R}$ in third equation are neglected, and from it one obtains

$$\frac{\partial u}{\partial z} = -\frac{u_r}{R}\frac{C_{13}\left(1-\frac{C_{12}}{C_{11}}\right)}{C_{33}-\frac{C_{13}^2}{C_{11}}} + \frac{e_{13}-\frac{e_{31}}{C_{11}}-\frac{e_{15}}{C_{44}}\left(C_{33}-\frac{C_{13}^2}{C_{11}}\right)}{C_{33}-\frac{C_{13}^2}{C_{11}}}k\phi_0 \sin kz. \quad (3.6)$$

Substituting (3.1), (3.2), (3.6) in last equation (1.3) one obtains

$$\phi_0'' + \frac{\phi_0'}{r} - \nu_0^2\phi_0 = \frac{e_{33}k^3A(r-R) + e_{31}\frac{kA}{r} - e_{33}\frac{kA}{R}C_{13}\frac{1-\frac{C_{12}}{C_{11}}}{C_{33}-\frac{C_{13}^2}{C_{11}}}}{\frac{e_{15}^2}{C_{44}}+\varepsilon_{11}}, \quad (3.7)$$

where

$$\nu_0^2 = k^2\frac{\varepsilon_{33}+e_{33}\frac{e_{15}}{C_{44}} + \frac{e_{33}-\frac{e_{31}}{C_{11}}C_{12}-e_{15}\frac{C_{33}-\frac{C_{13}^2}{C_{11}}}{C_{44}}}{C_{33}-\frac{C_{13}^2}{C_{11}}}}{\frac{e_{15}^2}{C_{44}}+\varepsilon_{11}}. \quad (3.8)$$

To simplify (3.7) one can assume that in terms with piezoelectric effects one can neglect terms with $\frac{1}{R}$ and one obtains equations

$$\phi_0'' - \nu_0^2\phi_0 = \frac{e_{33}k^3A(r-R)}{\frac{e_{15}^2}{C_{44}}+\varepsilon_{11}}. \quad (3.9)$$

For solution of (3.9) one obtains

$$\phi_0(r) = C_1 ch\nu_0(r-R) + C_2 sh\nu_0(r-R) - \chi A(r-R). \quad (3.10)$$

For solution out of shell for potential $\bar{\varphi}$ one obtains

$$\begin{aligned} r > R+h,\ \bar{\varphi} &= \cos kz K_0(kr)\bar{\phi}_+ e^{-i\omega t} + c.c., \\ r < R-h,\ \bar{\varphi} &= \cos kz I_0(kr)\bar{\phi}_- e^{-i\omega t} + c.c. \end{aligned} \quad (3.11)$$

For induction in shell D_r in (1.1) one obtain

$$D_r = -\phi_0'(r)\left(\frac{e_{15}^2}{C_{44}}+\varepsilon_{11}\right)\cos kz + c.c.$$

From continuity conditions for $r = R \pm h$ of $\varphi = \bar{\varphi}$, $D_r = \dfrac{\partial \varphi}{\partial r}$ one obtains

$$C_1 = 0,\ C_2 = \frac{\chi A}{\nu_0}, \tag{3.12}$$

$$\phi_0 = \frac{\chi A}{\nu_0} sh\nu_0 (r - R) - \chi A (r - R),\ \phi_0 \approx \frac{\chi A}{6} \nu_0^2 (r - R)^2 . \tag{3.13}$$

From (3.4), (3.5), (3.6), (3.12) one obtains

$$\begin{gathered} \mathrm{M} = \left(C_{33} - \frac{C_{13}^2}{C_{11}}\right) \sin kz k^2 A \frac{2h^3}{3}, \\ \int\limits_{R-h}^{R+h} \sigma_{\vartheta\vartheta} dr = \frac{\sin kz}{R} \left(1 - \frac{C_{12}^2}{C_{11}^2}\right) C_{11} A 2h - \\ -\left(C_{13} - \frac{C_{13} C_{12}}{C_{11}}\right) \frac{1}{R} A \sin kz 2h C_{13} \frac{1 - \dfrac{C_{12}}{C_{11}}}{C_{33} - \dfrac{C_{13}^2}{C_{11}}} \end{gathered} \tag{3.14}$$

where is used that function ϕ_0 is add with respect to $r - R$, and values of highly order smallness on are dropped out. Substituting of (3.14) in (3.5) one obtains

$$\rho\omega^2 = \left(C_{33} - \frac{C_{13}^2}{C_{11}}\right) k^4 \frac{h^2}{3} + \frac{1}{R^2} \left(C_{11} - \frac{C_{12}^2}{C_{11}} - \frac{\left(C_{13} - C_{13}\dfrac{C_{12}}{C_{11}}\right)^2}{C_{33} - \dfrac{C_{13}^2}{C_{11}}} \right) \tag{3.15}$$

and using also values (1.21),

$$\nu^2 = \frac{1}{4} k^2 h^2 + \frac{2}{3} \frac{1}{R^2} \tag{3.16}$$

which in the main order coincides with dispersion relation for elastic anisotropic shell. The numerical results by (3.16) are given in table 2 calculated by Kirchhoff hypothesis

Table 2

k	0.1	0.2	0.3	0.4	0.5
	0.000957427	0.00216025	0.00457347	0.00804156	0.0125266

The comparison of table 1 and table 2 shows that the results by space treatment are quite different form those obtained by hypothesis.

4. Calculations of frequencies by exact solution for piezoelectric plate

For piezoelectric strip equations of motion

$$\frac{\partial u_x}{\partial x^2}+\mu_1\frac{\partial^2 u_x}{\partial z^2}+e^2u_x+\mu_2\frac{\partial^2 u_z}{\partial x\partial z}+\frac{\partial^2\varphi}{\partial x\partial z}=0,$$
$$\mu_2\frac{\partial^2 u_z}{\partial x\partial z}+\mu_1\frac{\partial^2 u_z}{\partial x^2}+\mu_4\frac{\partial^2 u_z}{\partial z^2}+e^2u_z+\mu_5\frac{\partial^2\varphi}{\partial z^2}+\mu_6\frac{\partial^2\varphi}{\partial x^2}=0, \quad (4.1)$$
$$-\left(k_1^2\frac{\partial^2 u_x}{\partial x\partial z}+k_2\frac{\partial^2 u_z}{\partial x^2}+k_3\frac{\partial^2 u_z}{\partial z^2}\right)+\mu_7^2\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial z^2}=0.$$

For considered antisymmetric problem one has

$$u_x(x,z)=U_j sh\lambda_j pz\cos px,$$
$$u_z(x,z)=V_j ch\lambda_j pz\sin px, \quad (4.2)$$
$$\varphi(x,z)=\phi ch\lambda_j pz\sin px,$$

where is carried out on j summation from 1 to 3, Substituting of (4.2) in (4.1) for U_j, V_j, ϕ one obtain homogeneous system, where determinant

$$\det\left\|a_{ij}\right\|=0 \quad (4.3)$$

determining λ,

$$a_{11}=1-\mu_1\lambda^2-\nu^2,\ a_{12}=-\mu_2\lambda,\ a_{21}=a_{12},\ a_{13}=-\lambda,$$
$$a_{22}=-\mu_1+\mu_4\lambda^2+\nu^2,\ a_{23}=\mu_5\lambda^2-\mu_6,\ a_{31}=k_1^2\lambda,$$
$$a_{32}=k_2-k_3\lambda^2,\ a_{33}=\lambda^2-\mu_7^2.$$

One can write (4.2) in form [11]

$$u_x(x,z)=\alpha_j sh\lambda_j pzU_j\cos px,$$
$$u_z(x,z)=\beta_j ch\lambda_j pzU_j\sin px, \quad (4.4)$$
$$\varphi(x,z)=\gamma_j ch\lambda_j pzU_j\sin px,$$

where is carried out summation on j from to 3,

$$\alpha_j\left(\lambda_j\right)=a_{12}a_{23}-a_{13}a_{22},$$
$$\beta_j\left(\lambda_j\right)=a_{21}a_{13}-a_{11}a_{23},$$
$$\gamma_j\left(\lambda_j\right)=a_{11}a_{22}-a_{12}^2,$$

potential of electric field $\bar{\varphi}$ in region out of plate $|z|>h$ can be written as

$$\bar{\varphi}=\bar{\phi}e^{\mp pz}\sin px, \quad (4.5)$$

which satisfy the equation $\frac{\partial^2\overline{\varphi}}{\partial x^2}+\frac{\partial^2\overline{\varphi}}{\partial z^2}=0$. The stress components in plate are [11]

$$\begin{aligned}\sigma_{xx}&=C_{11}pt_j^*U_j sh\lambda_j pz\sin px,\\ \sigma_{zz}&=C_{33}pm_j^*U_j sh\lambda_j pz\sin px,\\ \sigma_{xz}&=C_{44}pn_j^*U_j ch\lambda_j pz\cos px,\end{aligned}\tag{4.6}$$

where there is summation on j from 1 to 3,

$$t_j^*=-\alpha_j+(\mu_2-\mu_1)\beta_j\lambda_j+(1-\mu_6)\gamma_j\lambda_j,$$

$$n_j^*=\alpha_j\lambda_j+\beta_j+\frac{\mu_6}{\mu_1}\gamma_j,\ \mu_8=\frac{C_{13}}{C_{33}},$$

$$m_j^*=-\mu_8\alpha_j+\tfrac{\mu_5}{\mu_4}\gamma_j\lambda_j+\beta_j\lambda_j.$$

Here there is no summation.
Boundary conditions $\sigma_{zz}(x,\pm h)=0,\ \sigma_{xz}(x,\pm h)=0$ are satisfied by

$$\begin{aligned}&U_j=\Delta_jU_0,\ \Delta_1=m_2n_3-m_3n_2\\ &\Delta_2=m_3n_1-m_1n_3,\ \Delta_1=m_1n_2-m_2n_1,\\ &m_j=m_j^*sh\lambda_jph,\ n_j=n_j^*ch\lambda_jph.\end{aligned}\tag{4.7}$$

From (4.4)-(4.7) and conditions $z=\pm h,\ \varphi=\overline{\varphi}$ one obtains

$$e^{-ph}\overline{\phi}=\gamma_j^*\Delta_jU_0,\ \gamma_j^*=\gamma_jch\lambda_jph.\tag{4.8}$$

Using also conditions of continuity z component of induction $z=\pm h$,

$$\begin{aligned}&D_z=\overline{D}_z,\ z=\pm h,\ \overline{D}_z=S\frac{\partial\overline{\varphi}}{\partial z},\\ &D_z=pq_j\Delta_jsh\lambda_jpzU_0\sin px,\\ &\overline{D}_z=\pm p\gamma_j^*\Delta_jU_0e^{\mp z}\sin px.\end{aligned}\tag{4.9}$$

One obtains the dispersion equation

$$\begin{aligned}&R_{21}(p,\nu)=0,\ R_{21}=R_2-\frac{S}{S_{33}}R_1,\\ &\begin{Bmatrix}R_1\\ R_2\end{Bmatrix}=\begin{Bmatrix}\gamma_j^*\\ q_j^*\end{Bmatrix}\Delta_j,\end{aligned}\tag{4.10}$$

where is carried out summation on j from 1 to 3,

$$\gamma_j^*=\gamma_jch\lambda_jph,\ q_j^*=q_jsh\lambda_jph,$$

$$q_j=-\gamma_j\lambda_j-e_{31}\mu_9\alpha_j+e_{33}\mu_9\beta_j\lambda_j,$$

$$\mu_9=\frac{e_{31}+e_{15}}{C_{11}S_{33}},$$

S_{33} is dielectric constant for plate $\frac{S}{S_{33}}<1$.

5. Piezoelectric case based on Kirchhoff hypothesis

One can obtain solution for piezoelectric plate with normal polarization based on Kirchhoff hypothesis. Equations of motion and of elastic induction are

$$\begin{aligned}&\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xz}}{\partial z}=\rho\frac{\partial^2 u_x}{\partial t^2},\\&\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{zz}}{\partial z}=\rho\frac{\partial^2 u_z}{\partial t^2},\\&\frac{\partial D_x}{\partial x}+\frac{\partial D_z}{\partial z}=0.\end{aligned}\tag{5.1}$$

From [11] these equations can be written as

$$\begin{aligned}&\frac{\partial^2 u_x}{\partial x^2}+\mu_1\frac{\partial^2 u_x}{\partial z^2}+e^2u_x+\mu_2\frac{\partial^2 u_z}{\partial x\partial z}+\frac{\partial^2\varphi}{\partial x\partial z}=0,\\&\mu_2\frac{\partial^2 u_z}{\partial x\partial z}+\mu_1\frac{\partial^2 u_z}{\partial x^2}+\mu_4\frac{\partial^2 u_z}{\partial z^2}+e^2u_z+\mu_5\frac{\partial^2\varphi}{\partial z^2}+\mu_6\frac{\partial^2\varphi}{\partial x^2}=0,\\&-\left(k_1^2\frac{\partial^2 u_x}{\partial x\partial z}+k_2\frac{\partial^2 u_z}{\partial x^2}+k_3\frac{\partial^2 u_z}{\partial z^2}\right)+\mu_7^2\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial z^2}=0,\end{aligned}\tag{5.2}$$

$$\begin{aligned}&\mu_1=\frac{C_{44}}{C_{11}},\ \mu_2=\frac{C_{13}+C_{44}}{C_{11}},\ \mu_4=\frac{C_{33}}{C_{11}},\ \mu_5=\frac{e_{33}}{e_{31}+e_{15}},\\&\mu_6=\frac{e_{15}}{e_{31}+e_{15}},\ \mu_7^2=\frac{S_{11}}{S_{33}},\ k_1^2=\frac{(e_{31}+e_{15})^2}{C_{11}S_{33}},\ k_2=\mu_6k_1^2,\\&k_3=\mu_5k_1^2,\ e^2=\frac{\omega^2\rho}{C_{11}}.\end{aligned}\tag{5.3}$$

Comparison of (5.1), (5.3) yields

$$\begin{aligned}&\sigma_{xx}=C_{11}\frac{\partial u_x}{\partial x}+C_{13}\frac{\partial u_z}{\partial z}+C_{11}\frac{\partial\varphi}{\partial z}\frac{e_{31}}{e_{31}+e_{15}},\\&\sigma_{zz}=C_{33}\frac{\partial u_z}{\partial z}+C_{13}\frac{\partial u_x}{\partial x}+C_{11}\mu_5\frac{\partial\varphi}{\partial z},\\&\sigma_{xz}=C_{44}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right)+C_{11}\mu_6\frac{\partial\varphi}{\partial x},\\&D_z=-\frac{\partial\varphi}{\partial z}+e_{31}\frac{e_{31}+e_{15}}{C_{11}S_{33}}\frac{\partial u_x}{\partial x}+e_{33}\frac{e_{31}+e_{15}}{C_{11}S_{33}}\frac{\partial u_z}{\partial z},\\&D_x=-\mu_7^2\frac{\partial\varphi}{\partial x}+e_{15}\frac{e_{31}+e_{15}}{C_{11}S_{33}}\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right),\end{aligned}\tag{5.4}$$

φ is connected with electrical potential by $\varphi=\frac{e_{31}+e_{15}}{C_{11}}\bar{\varphi}$. Due to Kirchhoff hypothesis one has

$$u_z\approx u_z(x),\ \sigma_{xz}\approx 0,\ \sigma_{zz}\approx 0.\tag{5.5}$$

From (5.4), (5.5) one can obtain relations

$$\begin{aligned}
&\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}=-\frac{C_{11}}{C_{44}}\mu_6\frac{\partial\varphi}{\partial x},\\
&\frac{\partial u_z}{\partial z}=-\frac{C_{13}}{C_{33}}\frac{\partial u_x}{\partial x}-\frac{C_{11}}{C_{33}}\mu_5\frac{\partial u_x}{\partial z},\\
&D_x=-\frac{\partial\varphi}{\partial x}\left(\mu_7^2+\frac{e_{15}^2}{C_{44}}\right),\\
&D_z=-\frac{\partial\varphi}{\partial z}\left(1+\frac{e_{33}^2}{C_{33}}\right)+\left(e_{31}-\frac{C_{13}}{C_{33}}e_{33}\right)\frac{e_{31}+e_{15}}{C_{11}S_{33}}\frac{\partial u_x}{\partial x}.
\end{aligned}\tag{5.6}$$

One can look for u_z and φ in form

$$u_z=A\cos px,\ \ \varphi=\cos px\phi_0'(z)\,.\tag{5.7}$$

Then (5.6) yields

$$\begin{aligned}
&u_x=zAp\sin px+\frac{C_{11}}{C_{44}}\mu_6 p\sin px\phi_0(z)\,,\\
&\frac{\partial u_z}{\partial z}=-\frac{C_{13}}{C_{33}}zAp^2\cos px-\frac{C_{13}}{C_{33}}\frac{C_{11}}{C_{44}}\mu_6p^2\cos px\phi_0-\frac{C_{11}}{C_{33}}\mu_5\cos px\phi_0'',\\
&D_z=-\cos px\phi_0''\left(1+\frac{e_{33}^2}{C_{33}}\right)+\\
&+\left(e_{31}-\frac{C_{13}}{C_{33}}e_{33}\right)\left(\frac{e_{31}+e_{15}}{C_{11}S_{33}}zA+\frac{e_{15}}{C_{44}S_{33}}\phi_0(z)\right)p^2\cos px.
\end{aligned}\tag{5.8}$$

Substituting (5.7), (5.8) into (5.2) gives

$$\begin{aligned}
&-k_1^2p^2A-k_1^2p^2\frac{C_{11}}{C_{44}}\mu_6\phi_0'+Ap^2k_2+Ap^2k_3\frac{C_{13}}{C_{33}}+\\
&+\frac{C_{13}}{C_{33}}\frac{C_{11}}{C_{44}}\mu_6p^2\phi_0'k_3+\frac{C_{11}}{C_{33}}\mu_5\phi_0'''k_3-\mu_7^2p^2\phi_0'+\phi_0'''=0,
\end{aligned}$$

or denoting $\phi'=\Phi$

$$\begin{aligned}
&\Phi''-\nu_1^2\Phi=A\zeta p^2,\\
&\nu_1^2=\frac{\mu_7^2+\frac{C_{11}}{C_{44}}\mu_6\left(k_1^2-k_3\frac{C_{13}}{C_{33}}\right)}{1+k_3\frac{C_{11}}{C_{33}}\mu_5}p^2,\ \ \zeta=\frac{k_1^2-k_2-k_3\frac{C_{13}}{C_{33}}}{1+k_3\frac{C_{11}}{C_{33}}\mu_5}.
\end{aligned}\tag{5.9}$$

The general solution of (5.9) yields

$$\Phi=C_1ch\nu_1z-\frac{\zeta p^2A}{\nu_1^2}.\tag{5.10}$$

Substituting (5.10) in (5.8) gives

$$D_z = -C_1\nu_1 \cos px\left(1+\frac{e_{33}^2}{C_{33}}\right) sh\nu_1 z+$$
$$+\left(e_{31}-\frac{C_{13}}{C_{33}}e_{33}\right)\frac{e_{31}+e_{15}}{C_{11}S_{33}} zAp^2\cos px+ \quad (5.11)$$
$$+\left(e_{31}-\frac{C_{13}}{C_{33}}e_{33}\right)\left(C_1\frac{sh\nu_1 z}{\nu_1}-\frac{\zeta^2 pA}{\nu_1^2}z\right)\frac{e_{15}}{C_{44}S_{33}}p^2\cos px.$$

Boundary conditions on $z = h$ give

$$\varphi = \bar{\varphi},\ D_z = \bar{D}_z. \quad (5.12)$$

Where for dielectric out of plate

$$\bar{\varphi} = C_3 e^{-p(z-h)}\cos px,\ \bar{D}_z = -C_3\frac{S}{S_{33}}pe^{-p(z-h)}\cos px. \quad (5.13)$$

On account of (5.7), (5.8) and (5.10)–(5.13) one obtains

$$z = h,$$
$$C_1 ch\nu_1 h - \frac{\zeta p^2 A}{\nu_1^2} = C_3,$$
$$-C_1\nu_1\left(1+\frac{e_{33}^2}{C_{33}}\right)sh\nu_1 h+\left(e_{31}-\frac{C_{13}}{C_{33}}e_{33}\right)\frac{e_{31}+e_{15}}{C_{11}S_{33}}hAp^2+ \quad (5.14)$$
$$+\left(e_{31}-\frac{C_{13}}{C_{33}}e_{33}\right)\left(C_1\frac{sh\nu_1 h}{\nu_1}-\frac{\zeta p^2 A}{\nu_1^2}h\right)\frac{e_{15}}{C_{44}S_{33}}p^2 = -C_3\frac{S}{S_{33}}p.$$

From (5.14) it follows that

$$C_1 sh\nu_1 h\left\{-\nu_1\left(1+\frac{e_{33}^2}{C_{33}}\right)+\left(e_{31}-\frac{C_{13}}{C_{33}}e_{33}\right)\frac{e_{15}}{C_{44}S_{33}}\frac{p^2}{\nu_1}\right\}+$$
$$+hAp^2\left(e_{31}-\frac{C_{13}}{C_{33}}e_{33}\right)\left(\frac{e_{31}+e_{15}}{C_{11}S_{33}}-\frac{e_{15}}{C_{44}S_{33}}\frac{\zeta p^2}{\nu_1^2}\right)= \quad (5.15)$$
$$=-\frac{S}{S_{33}}p\left(C_1 ch\nu_1 h-\frac{\zeta p^2 A}{\nu_1^2}\right),$$

$$C_1 = -\frac{hAp^2\left(e_{31}-\frac{C_{13}}{C_{33}}e_{33}\right)\left(\frac{e_{31}+e_{15}}{C_{11}S_{33}}-\frac{e_{15}}{C_{44}S_{33}}\frac{\zeta p^2}{\nu_1^2}\right)+\frac{S}{S_{33}}\frac{\zeta p^3 A}{\nu_1^2}}{sh\nu_1 h\left\{-\nu_1\left(1+\frac{e_{33}^2}{C_{33}}\right)+\left(e_{31}-\frac{C_{13}}{C_{33}}e_{33}\right)\frac{e_{15}}{C_{44}S_{33}}\frac{p^2}{\nu_1}\right\}+\frac{S}{S_{33}}pch\nu_1 h}. \quad (5.16)$$

From (5.1), neglecting of $\rho\frac{\partial^2 u_x}{\partial t^2}$ and multiplying on z, after integration on z, one obtains

$$\frac{\partial\int_{-h}^{h} z\sigma_{xx}dz}{\partial x}+\int_{-h}^{h} z\frac{\partial\sigma_{xz}}{\partial z}dz = 0$$

and after integrating on z by part in second term one obtains

$$Q = \int_{-h}^{h} \sigma_{xz} dz, \ \frac{\partial M}{\partial x} = Q, \ M = \int_{-h}^{h} z\sigma_{xx} dz. \tag{5.17}$$

One account boundary conditions $\sigma_{xz} = \sigma_{xx} = 0$ one obtains after integration of second equation in (5.1)

$$\frac{\partial^2 M}{\partial x^2} = \frac{\partial Q}{\partial x} = 2sh\frac{\partial^2 u_z}{\partial t^2}. \tag{5.18}$$

From (5.4), (5.6) one obtains

$$\sigma_{xx} = \left(C_{11} - \frac{C_{13}^2}{C_{33}}\right)\frac{\partial u_x}{\partial x} + \frac{C_{11}}{e_{31}+e_{15}}\left(e_{31} - \frac{e_{33}C_{13}}{C_{33}}\right)\frac{\partial \varphi}{\partial z}$$

and substituting of (5.7), (5.8), (5.10) one obtains

$$\begin{gathered}\sigma_{xx} = \left(C_{11} - \frac{C_{13}^2}{C_{33}}\right)\left(zA + \frac{C_{11}}{C_{44}}\frac{e_{15}}{e_{31}+e_{15}}p\left(\frac{C_1}{\nu_1}sh\nu_1 z - \frac{\zeta p^2 A}{\nu_1^2}z\right)\right)\times \\ \times p\cos px + \frac{C_{11}}{e_{31}+e_{15}}\left(e_{31} - \frac{e_{33}C_{13}}{C_{33}}\right)C_1\nu_1 sh\nu_1 z\cos px.\end{gathered} \tag{5.19}$$

Substituting (5.19) in (5.17) one obtains

$$\begin{gathered}M = \int_{-h}^{h}\left\{\left(C_{11} - \frac{C_{13}^2}{C_{33}}\right)\left(z^2Ap^2 + \frac{C_{11}}{C_{44}}\frac{e_{15}p^2}{e_{31}+e_{15}}\left(\frac{C_1}{\nu_1}zsh\nu_1 z - \frac{\zeta p^2 A}{\nu_1^2}z^2\right)\right) + \right. \\ \left. + \frac{C_{11}}{e_{31}+e_{15}}\left(e_{31} - e_{33}\frac{C_{13}}{C_{33}}\right)C_1\nu_1 sh\nu_1 z\right\}dz,\end{gathered}$$

and on account that $\int_{-h}^{h} zsh\nu_1 zdz = \frac{2}{3}\nu_1 h^3$ one obtains

$$\begin{gathered}M = \frac{2h^3}{3}\cos px\left\{p^2\left(C_{11} - \frac{C_{13}^2}{C_{33}}\right)\left(\left(1 - \frac{\zeta p^2}{\nu_1^2}\frac{C_{11}}{C_{44}}\frac{e_{15}}{e_{31}+e_{15}}\right)A + \frac{C_{11}}{C_{44}}\frac{C_1 e_{15}}{e_{31}+e_{15}}\right) + \right. \\ \left. \frac{C_{11}C_1\nu_1^2}{e_{31}+e_{15}}\left(e_{31} - e_{33}\frac{C_{13}}{C_{33}}\right)\right\}, \ C_1 \approx \frac{\zeta p^2 A}{\nu_1^2}.\end{gathered}$$

In elastic case from (5.16) one obtains $C_1 = 0$,

$$M = \frac{2h^3p^2}{3}A\cos px\left(C_{11} - \frac{C_{13}^2}{C_{33}}\right).$$

Dispersion relation can be obtained from (5.18), thus one has

$$\frac{h^2p^4}{3}\left(C_{11} - \frac{C_{13}^2}{C_{33}}\right) + \frac{C_{11}}{e_{31}+e_{15}}\frac{h^2\zeta p^4}{3}\left(e_{31} - e_{33}\frac{C_{13}}{C_{33}}\right) = \rho\omega^2. \tag{5.20}$$

Values of constants for $BaTiO_3$ are

$$C_{11} = 1,5 * 10^{11} N/m^2,\ C_{13} = 6,6 * 10^{10} N/m^2,\ C_{33} = 1,4 * 10^{11} N/m^2,$$

$$C_{44} = 4,5 * 10^{10} N/m^2,\ S_{33} = 10^{-9} \phi/m,\ e_{31} = -4K/m^2,$$

$$e_{33} = 17K/m^2,\quad e_{15} = 11K/m^2.$$

Then

$$\frac{C_{13}}{C_{11}} = \frac{1}{2},\ \frac{C_{13}}{C_{33}} = \frac{1}{2},\ \frac{C_{11}}{C_{44}} = 3,\ \frac{C_{11}}{C_{33}} = 1,\ \mu_7^2 = 1,$$

$$\frac{e_{15}}{e_{31}} = -3,\ \frac{e_{33}}{e_{31}} = -4,\ \mu_6 = \frac{3}{2},\ \mu_5 = 2,\ k_1^2 = \frac{n}{300},\ h = 0,1cm,\ n = \overline{1,...,10.},$$

And from (5.20) one has

$$\nu = \frac{1}{2} h p^2 \sqrt{1 + \frac{3k_1^2}{1 + 4k_1^2}}. \tag{5.21}$$

The results of calculations by the exact treatment, made by (4.10), are brought in (4.3) tables 3, 4 and by hypothesis are done by (5.21) and are done by table 5.

Table 3

$$\frac{S_{33}}{S} = 10$$

$k_1^2 = \frac{n}{300}$	p				
	0.1	0.2	0.3	0.4	0.5
n=0.1	0.660153	0.660142	0.660132	0.660121	0.660111
n=1	0.696237	0.664531	0.716595	0.716833	0.717054
n=2	0.588632	0.588632	0.588632	0.588632	0.588632i
n=3	0.594389	0.503435	0.594389	0.594389	0.594389
n=4	0.720844	0.703199	0.733194	0.73344	0.733659
n=5	0.693198	0.693535	0.693854	0.694158	0.694447
n=10	0.720122	0.720582	0.720998	0.748948	0.72173

Table 4

$$\frac{S_{33}}{S} = 1$$

$k_1^2 = \frac{n}{300}$	p				
	0.1	0.2	0.3	0.4	0.5
n=0.1	0.588	0.57	0.5888	0.5889	0.589002
n=1	0.71	0.73	0.7369	0.782	0.782102
n=2	0.58	0.588	0.58863	0.58	0.5886
n=3	0.594	0.59	0.594	0.59	0.5943
n=4	0.73	0.73	0.736	0.73	0.0.7371
n=5	0.6	0.6i	0.6053	0.605	0.6054
n=10	0.63	0.63	0.630	0.6307i	0.63072

Table 5

The Kirchhoff case table $\frac{S_{33}}{S} = 10$

$k_1^2 = \frac{n}{300}$	p				
	0.1	0.2	0.3	0.4	0.5
n=0	0.005	0.01	0.015	0.02	0.025
n=0.1	0.00500416	0.0100083	0.0150125	0.0200166	0.0250208
n=1	0.00504095	0.0100819	0.0151229	0.0201638	0.0252048
n=2	0.00508052	0.010161	0.0152416	0.0203221	0.0254026
n=3	0.00511878	0.0102376	0.0153563	0.0204751	0.0255939
n=4	0.0051558	0.0103116	0.0154674	0.0206232	0.025779
n=5	0.00519164	0.0103833	0.0155749	0.0207666	0.0259582
n=10	0.00535504	0.0107101	0.0160651	0.0214202	0.0267752

Comparision of tables 3 and 5 shows that the solution by exact space treatment essentially is distinguished from solution obtained due to Kirchhoff hypothesis.

Conclusion

The derivation of disspersion relation for free bending vibrations of thin piezoelastic cylindrical shells with longitudinal polarization and for plates with normal polarization by exact space treatment, proposed at first for elastic plates by V. Novatski, is given. It is done numerical solutions of obtained transcendent equations. Also the same considerations are made by treatment based on Kirchhoff hypothesis.
The table 1 corresponds to shell with radius $R = 10^3 cm$ calculated by space treatment, and table 2 by hypothesis, in the last in main order frequency does not depend from piezoelectric properties. The results of table 1 and table 2 are quite different. Also are constructed by space treatment tables 3, 4 for piezoelectric plates and table 5 by averaged method based on hypothesis. The tables 3 and 5 are distinguished by several times. Thus in considered problem as in magnetoelastic plates and shells in piezoelectricity Kirchhoff hypothesis not applicable.

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ОПРЕДЕЛЕНИЕ ЛИНЕЙНЫХ ЧАСТОТ ИЗГИБНЫХ КОЛЕБАНИЙ ПЬЕЗОЭЛЕКТРИЧЕСКИХ ОБОЛОЧЕК И ПЛАСТИН ПО ТОЧНОМУ И ОСРЕДНЕННОМУ ПОДХОДАМ

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В работе рссматривается вывод и численное решение дисперсионных соотношений для частот изгибных свободных колебаний пьезоэлектрических цилиндрических тонких оболочек с продольной поляризацией и тонких пластин с поперечной поляризацией. Решение дается по точному пространственному подходу и по гипотезе Кирхгоффа. Сравнение полученных таблиц показывает, что частоты по точному и основанному на гипотезе Кирхгоффа подходам значительно различаются.

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