МЕХАНИКА

CONSTITUTIVE MODELS OF ANISOTROPIC DAMAGE AND MODELLING OF DAMAGING MICROPROCESSES IN SOLIDS

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A thermodynamic analysis of three-dimensional anisotropic damage state and its time evolution is presented in an attempt to obtain a deeper insight into damaging phenomena and find the canonical state parameters required for a damage state description. The analysis of damage state is based on the canonical hidden variable technique and developed in the two canonical variants — the energy and the entropy ones. Thermodynamic damage state potentials in their canonical forms are obtained. In the case of isothermal damaging the canonical net stress tensor is derived. A variant of the canonical description of damage providing its notion originating from irreversible thermodynamics canonical definition of directional damage variable is discussed. The canonical representations of the thermodynamic damage state potentials in terms of damage tensors are derived. Those involve the only metric invariant of a damage state — the canonical norm. Directional damage averaging the damage represented by the second and the fourth rank damage tensors is considered. In order to demonstrate a superiority of the canonical formalism a general thermodynamic analysis of brittle and ductile damaging processes is carried out by the canonical technique. Universal equations of damage balance in the course of damage growth in solids are obtained.

1. Introduction

Damage is the decrease in elasticity property consequent on a decrease of the areas that transmit internal forces, through the appearance and subsequent growth of microcavities and microcracks [1].

Accurate analytical description of damage is a complicated problem, which has generated a great deal of controversy. Much efforts have been devoted to the refinements of the phenomenological theory of damage. Continuum damage mechanics is supported by the general concepts of irreversible thermodynamics and the formalism of hidden state variables [1], [2]. A unified thermomechanical framework of which the physical description requires the consideration of additional variables of state and of their gradients in order to account for marked damage localization effects is presented in [3]. A general thermodynamic

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model of three-dimensional anisotropic damage state based on the canonical formalism has been discussed in [4], [5].

After the Introduction, the paper includes a short review of the canonical formalism (Section 2) with the objective of establishing a reference framework of concepts, notations and terminology, and derivation the specific free energy state potential of damage from the canonical asymptotics [4].

In Section 3 a variant of the canonical description of three-dimensional anisotropic damage state is presented. As it is shown the canonical hidden state variables provide a definition of the directional damage variable. Under such a definition the canonical damage state variables can be interpreted at least asymptotically as the Fourier coefficients of the expansion of the directional damage variable into the sum of the Laplace spherical harmonics. Thus, the canonical norm, known from our previous discussion [4], which is involved in the expressions of thermodynamic damage state potentials can be estimated by the classical Fourier–Bessel formula.

An anisotropic damage state is often represented by a tensor state variable. We had to reestablish the definition of the second rank damage tensor in order to elucidate its origin from the classical Finger strain tensor [4]. If necessary, following [6] one can define higher rank damage tensors by polynomial in the director components approximations of the directional damage variable. Directional damage averaging the damage represented by the second and the fourth rank damage tensors is considered in Section 3.

The discussion is then proceeded to a general thermodynamic analysis of brittle and ductile damaging processes in solids (Section 4). Advantages of the canonical technique can be learned from that part of the discussion.

2. Thermodynamic state potentials of damage

Our choice of the thermodynamic basis for the description of damage state is determined by the advantages of the canonical formalism developed in the previous discussions [4], [5].

Let ξ_β denote the canonical entropy hidden variables, whereas ς_β — the canonical energy variables. As it is shown [4], the specific entropy s_D and the specific internal energy u_D of a damage state can be obtained as

$$s_D = \frac{2s_0(u_0, \mathbf{Y}^e)}{\sigma^2 \, ||\xi||^2}, \quad u_D = -\frac{2u_0(s_0, \mathbf{Y}^e)}{\Delta^2 \, ||\varsigma||^2}, \tag{2.1}$$

where $s_0 = s_0(u_0, \mathbf{Y}^e)$, $u_0 = u_0(s_0, \mathbf{Y}^e)$ are the specific entropy and the specific internal energy in the undamaged state, $\mathbf{Y}^e$ is the elastic part of the strain tensor $\mathbf{Y}$, $\sigma = \sigma(u_D, \mathbf{Y}^e)$, $\Delta = \Delta(s_D, \mathbf{Y}^e)$. In order to represent stress-strain state of a damaged element we employ a pair of the tensors $\mathbf{Y}, \mathbf{S}$, where $\mathbf{Y}$ is the strain tensor, $\mathbf{S}$ is the work-conjugated stress tensor:

$$\operatorname{tr}(J\mathbf{TD}) = \operatorname{tr}(\mathbf{S}\dot{\mathbf{Y}}),$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\mathbf{D}$ is the strain rate tensor (the symmetrized spatial velocity gradient), J is the scalar defined by the Jacobian of the deformation.

In the case of isothermal damaging it is expedient to introduce the specific Helmholtz free energy of damage state ψ_D:

$$\psi_D(\vartheta_\varsigma, \mathbf{Y}^e, \|\boldsymbol{\varsigma}\|) = u_D(s_D, \mathbf{Y}^e, \|\boldsymbol{\varsigma}\|) - \vartheta_\varsigma s_D, \tag{2.2}$$

$$\vartheta_\varsigma = \left(\frac{\partial u_D}{\partial s_D}\right)_{\mathbf{Y}^e, \varsigma_\gamma}, \tag{2.3}$$

where ϑ_ς is the absolute thermodynamic temperature in the canonical energy ς-representation and the specific entropy expressed via the variables $\vartheta_\varsigma, \mathbf{Y}^e, \|\boldsymbol{\varsigma}\|$ by equation (2.3) needs to be substituted in (2.2): $s_D = g(\gamma, \mathbf{Y}^e)$, $\gamma = \vartheta_\varsigma \|\boldsymbol{\varsigma}\|^2$.

Thus the specific free energy is obtained as

$$\psi_D = \vartheta_\varsigma \left[\frac{f(\gamma, \mathbf{Y}^e)}{\gamma} - g(\gamma, \mathbf{Y}^e)\right], \quad \gamma\frac{\partial g}{\partial \gamma} = \frac{\partial f}{\partial \gamma}.$$

Assuming a significant entropy increase in the course of isothermal damaging, we adopt the following asymptotics

$$\frac{\gamma^h g(\gamma, \mathbf{Y}^e)}{G(\mathbf{Y}^e)} = 1 + o(1) \tag{2.4}$$

as $\gamma \to 0$, where h is a constant, $G(\mathbf{Y}^e)$ is a function of the elastic strain tensor.

Additionally assuming that $h = 1/2$, which accords to the effective stress concept of continuum damage mechanics, one can then derive

$$\psi_D = -\frac{2\sqrt{\vartheta_\varsigma} G(\mathbf{Y}^e)}{\|\boldsymbol{\varsigma}\|}. \tag{2.5}$$

After that the work-conjugated stress tensor can be obtained as

$$\mathbf{S}_\varsigma = -\frac{2\rho_R \sqrt{\vartheta_\varsigma}}{\|\boldsymbol{\varsigma}\|}\frac{\partial G(\mathbf{Y}^e)}{\partial \mathbf{Y}^e}, \tag{2.6}$$

where ρ_R is the mass density at the reference state, the symbol ς in the subscripts refers to the energy variant of the canonical formalism. As the damage grows, the canonical energy norm $\|\boldsymbol{\varsigma}\|$ tends to zero and the work-conjugated stress is increasing. This increase is a manifestation of the magnifying stress effect caused by progressive damaging. It should be also noted that such a decomposition of the stress tensor into a magnifying factor and an elastic stress tensor is only valid as an isothermal asymptotics of the work-conjugated stresses represented in the canonical thermodynamic basis and is afforded by the energy canonical asymptotics.

In the canonical entropy ξ-representation the work-conjugated stress tensor $\mathbf{S}_\xi$ can be determined as follows

$$\mathbf{S}_\xi = -\rho_R \left\{\left(\frac{\partial s_D}{\partial u_D}\right)_{\mathbf{Y}^e, \xi_\gamma}\right\}^{-1} \left(\frac{\partial s_D}{\partial \mathbf{Y}^e}\right)_{u_D, \xi_\beta}.$$

3. A variant of the canonical description of damage

It is known [4], that the canonical description of damage is not unique. Any realization of the canonical description does not change the form of the canonical asymptotics (2.1) and the canonical norms $\|\xi\|$, $\|\varsigma\|$.

The specific free energy canonical asymptotics (2.5) involves the canonical norm $\|\varsigma\| = \sqrt{\varsigma_\beta \varsigma_\beta}$, where ς_β are the canonical energy hidden variables [4]. An adequate description of anisotropic damage state requires an appropriate directional damage variable $\varsigma = \varsigma(\mathbf{n})$, where $\mathbf{n}$ is a unit three-dimensional vector often referred to as a director. In the actual damaged state the value of ς associated with the director $\mathbf{n}$ is the damage measured in some way.

Let $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$ be a local orthonormal basis in the space. By introducing the angle coordinates Θ and Φ on the sphere of unit directions with respect to the coordinate axes determined by the vectors $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$, we then can consider the orthogonal set of the Laplace spherical harmonics $Y_l^{(k)}(\Theta, \Phi)$ $(0 \leqslant \Theta \leqslant \pi,$ $0 \leqslant \Phi \leqslant 2\pi)$ consisting of $2l + 1$ functions:

$$Y_l^{(k)}(\Theta, \Phi) = i^{-k} \sqrt{\frac{(l-k)!}{(l+k)!}}\, P_l^{(k)}(\cos\Theta) e^{-ik\Phi} \qquad (k = -l, ..., l),$$

where

$$P_l^{(-m)}(\cos\Theta) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^{(m)}(\cos\Theta) \quad (m = 1, 2, ..., l),$$

$P_l^{(0)} = P_l$ are the Legendre polynomials defined as

$$P_l(z) = \frac{1}{2^l l!} \frac{d^l}{dz^l} \left[(z^2 - 1)^l \right],$$

$P_l^{(k)}$ are the associated Legendre functions

$$P_l^{(k)}(z) = (1 - z^2)^{\frac{k}{2}} \frac{d^k}{dz^k} P_l(z).$$

The identities $Y_l^{(-m)} = (-1)^m Y_l^{(m)*}$ $(l = 0, 1, ...;\ m = 1, 2, ..., l)$ are valid. Hereafter, the asterisk denotes the complex conjugate. The spherical harmonics constitute a complete system of orthogonal on the unit sphere functions.

The system

$$\tilde{Y}_l^{(k)} = i^k \sqrt{2l+1}\, Y_l^{(k)} \quad (l = 0, 1, \ldots;\ k = -l, \ldots l)$$

is a orthonormal basis on the unit sphere. This means that any directional damage variable $\varsigma = \varsigma(\mathbf{n})$ can be expanded into the Fourier series

$$\varsigma(\Theta, \Phi) = \sum_{l=0}^{\infty} \sum_{k=-2l}^{2l} c_{2l}^{(k)} \tilde{Y}_{2l}^{(k)}(\Theta, \Phi), \tag{3.1}$$

where $c_{2l}^{(k)}$ are the Fourier coefficients

$$c_{2l}^{(k)} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \varsigma(\Theta, \Phi) \tilde{Y}_{2l}^{(k)*}(\Theta, \Phi) \sin\Theta d\Theta d\Phi. \tag{3.2}$$

The root-mean-square and the average of the directional damage are determined by

$$\bar{\varsigma} = \sqrt{\sum_{l=0}^{\infty} \sum_{k=-2l}^{2l} \left|c_{2l}^{(k)}\right|^2}, \quad <\varsigma>= c_0^{(0)}. \tag{3.3}$$

The root-mean-square of the directional damage variable has the similar to the canonical norm $||\boldsymbol{\varsigma}||$ expansion in the sum of the squared parameters. Thus, a variant of the canonical energy description is obtained by the definition of the canonical directional damage variable $\varsigma = \varsigma(\mathbf{n})$ according to equation (3.1), wherein

$$\sum_{l=0}^{\infty} \sum_{k=-2l}^{2l} \left|c_{2l}^{(k)}\right|^2 = \sum_{\beta} \varsigma_\beta^2, \quad ||\varsigma||^2 \frac{\partial \ln |\psi_D|}{\partial \varsigma_\beta} = -\varsigma_\beta. \tag{3.4}$$

The canonical directional damage variable $\varsigma = \varsigma(\mathbf{n})$ can be then incorporated in an alternative damage description scheme by extracting damage tensors from the orientation distribution $\varsigma = \varsigma(\mathbf{n})$. Those afterwards can be employed for tensor representations of anisotropic damage state. Extracting damage tensors from the canonical orientation distribution can be realized by the technique proposed, for instance, in [6].

As it was elucidated in our previous discussion [7], the symmetric second rank damage tensor $\mathbf{D}$ could be defined by the reduction of the effective load carrying area of the plane element normal to director $\mathbf{n}$ according to the relation

$$\varsigma = \sqrt{\operatorname{tr}[(\mathbf{I} - \mathbf{D})^2 \mathbf{n} \otimes \mathbf{n}]}\,. \tag{3.5}$$

Symmetry of the damage tensor provides a clear mechanical interpretation for damage principal directions and values. In view of symmetry, the damage tensor can be represented in the spectral form

$$\mathbf{D} = \sum_{\alpha=1}^{3} D_{(\alpha)} \mathbf{d}_{(\alpha)} \otimes \mathbf{d}_{(\alpha)}, \tag{3.6}$$

wherein $\mathbf{d}_{(1)}$, $\mathbf{d}_{(2)}$, $\mathbf{d}_{(3)}$ are vectors of the orthonormal eigenbasis (in general, different from that introduced above), and $D_{(1)}$, $D_{(2)}$, $D_{(3)}$ — the damage tensor eigenvalues, called also as principal damages.

Substitution of spectral decomposition (3.6) into equation (3.5) gives

$$\varsigma = \sqrt{(1 - D_{(1)})^2 n_{(1)}^2 + (1 - D_{(2)})^2 n_{(2)}^2 + (1 - D_{(3)})^2 n_{(3)}^2}, \tag{3.7}$$

where $n_{(i)}$ are the components of the unit vector $\mathbf{n}$ with respect to the damage eigenbasis.

For the plane element, orthogonal to the principal axis of damage labelled by γ, we obtain from the latter equation the following formula

$$D_{(\gamma)} = \frac{dA_{(\gamma)} - dA^*_{(\gamma)}}{dA_{(\gamma)}} \quad \text{(no sum on } \gamma;\ \gamma = 1, 2, 3), \tag{3.8}$$

that accords to the classical Kachanov–Rabotnov definition of the damage variable.

If one renumber the principal damages in the following order

$$D_{(3)} \leqslant D_{(2)} \leqslant D_{(1)},$$

then for an arbitrary orientation $\mathbf{n}$ the following double sided estimation is derived:

$$D_{(3)} \leqslant 1 - \varsigma \leqslant D_{(1)}.$$

The principal damages can be represented in terms of the principal damage stretches $L^D_{(\alpha)}$ (see discussion in [4]) as follows

$$1 - D_{(\gamma)} = \frac{L^D_{(1)} L^D_{(2)} L^D_{(3)}}{L^D_{(\gamma)}},$$

or vice versa

$$\begin{aligned} L^D_{(1)} &= \sqrt{\frac{(1 - D_{(2)})(1 - D_{(3)})}{(1 - D_{(1)})}}, \; L^D_{(2)} = \sqrt{\frac{(1 - D_{(1)})(1 - D_{(3)})}{(1 - D_{(2)})}}, \\ L^D_{(3)} &= \sqrt{\frac{(1 - D_{(1)})(1 - D_{(2)})}{(1 - D_{(3)})}}. \end{aligned} \tag{3.9}$$

Consider then the mean of the directional damage variable ς as represented by the second order approximation (3.7). We shall use the notations θ, φ for the spherical angles. Let $C_{(j)} = (1 - D_{(j)})^2$ $(j = 1, 2, 3)$ and a pair of angular brackets $<>$ denote the averaging over the unit sphere. The following iterated integral

$$<\varsigma>= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \sqrt{C_{(1)} \sin^2\theta \cos^2\varphi + C_{(2)} \sin^2\theta \sin^2\varphi + C_{(3)} \cos^2\theta} \sin\theta d\theta d\varphi \tag{3.10}$$

by making use of the result [8] (formula 2.597.2) becomes

$$<\varsigma>= \frac{2}{\pi} \int_0^1 \mathbf{E}(k) \sqrt{(C_{(3)} - C_{(2)})\tau^2 + C_{(2)}}\, d\tau, \quad k = \sqrt{\frac{(1 - \tau^2)(C_{(2)} - C_{(1)})}{(C_{(3)} - C_{(2)})\tau^2 + C_{(2)}}}, \tag{3.11}$$

where $\mathbf{E}(k)$ is the complete elliptic integral of the second kind, k is the modulus.

The number of the independent parameters in (3.11) can be reduced by introducing ratios of the eigenvalues $C_{(1)}$, $C_{(2)}$, $C_{(3)}$:

$$\frac{\pi <\varsigma>}{2\sqrt{C}} = I(p_1, p_2) \qquad (p_1 = C_{(1)}/C_{(2)}, \; p_2 = C_{(2)}/C_{(3)}, \; C = C_{(3)}), \tag{3.12}$$

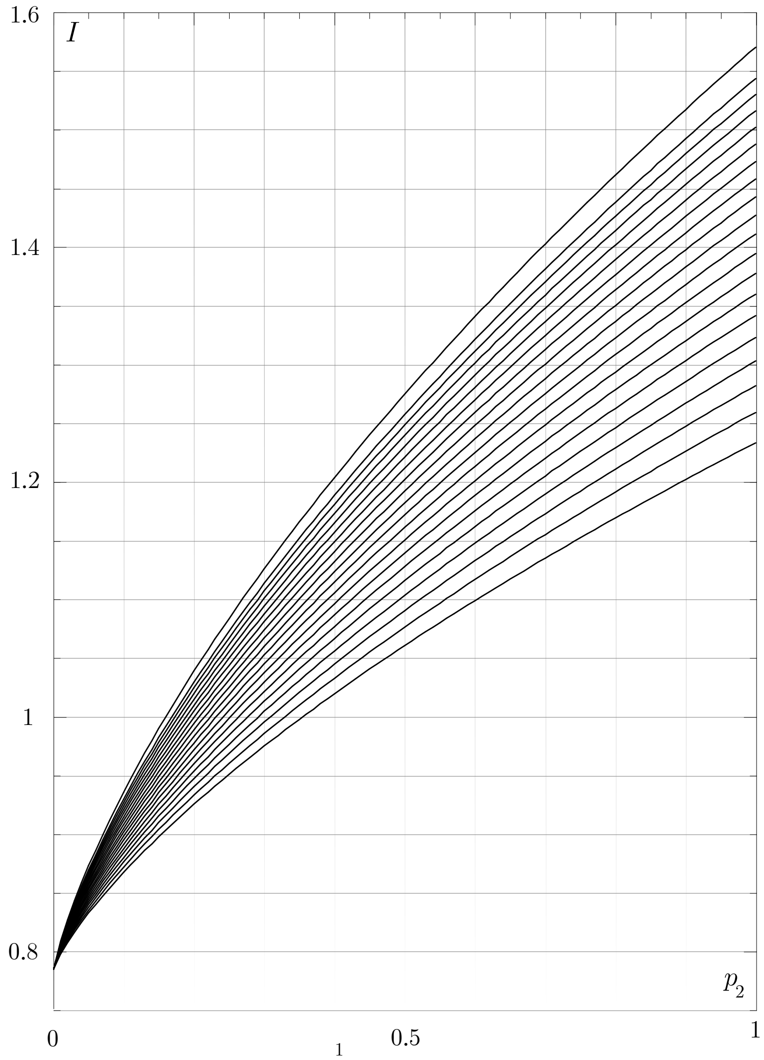
where

$$I(p_1, p_2) = \int_0^1 \sqrt{(1 - p_2)\tau^2 + p_2}\, \mathbf{E}\left(\sqrt{p_2(1 - \tau^2)(1 - p_1)((1 - p_2)\tau^2 + p_2)^{-1}}\right) d\tau.$$

The following expressions for the variables p_1, p_2, C in terms of the principal damage stretches (see equations (3.9)) are valid:

$$\sqrt{p_1} = \frac{L^D_{(2)}}{L^D_{(1)}}, \quad \sqrt{p_2} = \frac{L^D_{(3)}}{L^D_{(2)}}, \quad \sqrt{C} = L^D_{(1)} L^D_{(2)}. \tag{3.13}$$

The isolines $p_1(p_2, I) = \text{const}$ for values $p_1 = 0.0$, $p_1 = 0.05$, ... , $p_1 = 1.0$ are plotted in Figure.



$$I(p_1, 0) = \frac{\pi}{4}, \ I(0,1) = \int_0^1 \frac{k\mathbf{E}(k)dk}{\sqrt{1-k^2}} = 1.233701 ..., \ I(1,1) = \frac{\pi}{2}$$

Fig. The graphs of isolines $p_1(p_2, I) = \text{const}$

The mean of $I = I(p_1, p_2)$ is:

$$\langle I\rangle = \int_0^1 \int_0^1 I(p_1, p_2)\, dp_1 dp_2 = 1.157790\ldots . \qquad (3.14)$$

The average value (over the unit square) of the ratio $\langle\varsigma\rangle/\sqrt{C}$ is obtained as

$$\left\langle \frac{\langle\varsigma\rangle}{\sqrt{C}} \right\rangle = \frac{2}{\pi} \langle I\rangle = 0.737072\ldots . \qquad (3.15)$$

One more useful representation is:

$$\frac{\pi}{2} \frac{\langle\varsigma\rangle}{\sqrt{C}} = q^{-1}(1 + p^2 q^2) J(p, q),$$

where the improper integral

$$J(p, q) = \int_0^q \frac{k\mathbf{E}(k)\, dk}{\sqrt{q^2 - k^2}\, (1 + p^2 k^2)^2}$$

depends on the parameters

$$p = \sqrt{\frac{L_{(1)}^{D\,2} L_{(2)}^{D\,2} - L_{(1)}^{D\,2} L_{(3)}^{D\,2}}{L_{(1)}^{D\,2} L_{(3)}^{D\,2} - L_{(2)}^{D\,2} L_{(3)}^{D\,2}}}, \quad q = \sqrt{1 - \frac{L_{(2)}^{D\,2}}{L_{(1)}^{D\,2}}}.$$

A more symmetric formula for the involved ratio must be remarked:

$$\frac{\pi}{2} \frac{\langle\varsigma\rangle}{\sqrt{C}} = (1 + p^{*2}) J^*(p^*, q^*), \qquad (3.16)$$

wherein

$$J^*(p^*, q^*) = \int_0^1 \frac{k^* \mathbf{E}(q^* k^*)\, dk^*}{\sqrt{1 - k^{*2}} \left(1 + p^{*2} k^{*2}\right)^2}, \quad p^* = \sqrt{\frac{L_{(2)}^{D\,2}}{L_{(3)}^{D\,2}} - 1}, \quad q^* = \sqrt{1 - \frac{L_{(2)}^{D\,2}}{L_{(1)}^{D\,2}}}.$$

Integral J^* for weak damage anisotropy $(q^* \to 0)$ is obtained as

$$J^*(p^*, q^*) = \frac{\pi}{2p^{*4}} \left(1 + \frac{1}{4} q^{*2} p^{*-2}\right) \left\{ \frac{1}{4} (1 + p^{*-2})^{-3/2} \ln \frac{\sqrt{1 + p^{*-2}} + 1}{\sqrt{1 + p^{*-2}} - 1} + \frac{1}{2p^{*-2}(1 + p^{*-2})} \right\} - $$
$$- \frac{1}{8p^{*4}} \frac{q^{*2}}{\sqrt{1 + p^{*-2}}} \ln \frac{\sqrt{1 + p^{*-2}} + 1}{\sqrt{1 + p^{*-2}} - 1} + O(q^{*4}).$$

The latter equation provides the following exact formula for integral J^* in equation (3.16) if damage state is axial symmetric (i.e. $q^* = 0$):

$$J^*(p^*, 0) = \frac{\pi}{2p^{*4}} \left\{ \frac{1}{4} (1 + p^{*-2})^{-3/2} \ln \frac{\sqrt{1 + p^{*-2}} + 1}{\sqrt{1 + p^{*-2}} - 1} + \frac{1}{2p^{*-2}(1 + p^{*-2})} \right\}. \qquad (3.17)$$

The average value of the directional damage variable ς, approximated according to the equation

$$\varsigma^4 = \mathrm{tr}\,[\mathbf{n} \otimes \mathbf{n}\mathbf{C}\mathbf{n} \otimes \mathbf{n}] \quad (\sqrt[4]{\mathbf{C}} = \mathbf{I} - \mathbf{D}),$$

can be expressed analytically by using the elliptic functions of modulus $1/\sqrt{2}$, although the corresponding expression is rather complicated:

$$<\varsigma>= \frac{2}{\pi}\int_{u_1}^{u_2} \frac{\kappa'(u)}{1+\kappa^2(u)}du\left\{\frac{\beta_u}{8\sqrt[4]{\alpha_u^3}}\left[\frac{4\sqrt[4]{\alpha_u^3}\sqrt[4]{C_{(33)}}}{\psi_u} - \pi + \ln|\tau_u| + 2\mathrm{arctg}\sqrt[4]{\frac{C_{(33)}}{\alpha_u}}\right]+\right.$$
$$+\frac{u}{\sqrt{2}\sqrt[4]{\gamma_u}\sqrt[4]{\alpha_u\gamma_u-\beta_u^2}} + \frac{1}{\sqrt{\gamma_u}}(2\beta_u^2+\alpha_u\gamma_u)\left(\frac{1}{2\sqrt{\alpha_u}} - \frac{1}{4\sqrt[4]{\alpha_u}}\right)\times \tag{3.18}$$
$$\left.\times(f_1^+(u,\omega_u^+) - f_1^-(u,\omega_u^-)) + \frac{2\beta_u^2+\alpha_u\gamma_u}{4\sqrt{\gamma_u}}(f_2^+(u,\omega_u^+)+f_2^-(u,\omega_u^-))\right\},$$

wherein $\beta_u^2 - \alpha_u\gamma_u > 0$, $\tau_u = (\sqrt[4]{C_{(33)}} - \alpha_u)/(\sqrt[4]{C_{(33)}} - \sqrt[4]{\alpha_u})$, $\omega_u^\pm = 1/2 \pm \omega_u$,

$$f_1^\pm(u,\rho^2) = \frac{\sqrt[4]{C_{(33)}}}{\sqrt{2}\sqrt{\mathrm{dn}^{-2}u-1}(\sqrt{C_{(33)}^{-1}}(\mathrm{dn}^{-2}u-1) \pm \sqrt{\alpha_u})}\left[\Pi(u,\rho^2) - u\right],$$

$$f_2^\pm(u,\rho^2) = \frac{\sqrt[4]{C_{(33)}}}{\sqrt{2}\sqrt{\mathrm{dn}^{-2}u-1}(\sqrt{C_{(33)}^{-1}}(\mathrm{dn}^{-2}u-1) \pm \sqrt{\alpha_u})^2}\left\{\frac{1}{(\rho^2-1)(1-2\rho^2)}\times\right.$$
$$\left.\times\left[\rho^2E(u) + (\frac{1}{2}-\rho^2)u + (3\rho^2-\rho^4-\frac{3}{2})\Pi(u,\rho^2) - \frac{\rho^4\mathrm{sn}u\mathrm{cn}u\mathrm{dn}u}{1-\rho^2\mathrm{sn}u^2}\right] + u - 2\Pi(u,\rho^2)\right\},$$
$$\alpha_u = C_{(33)} + \frac{C_{(11)} - 2C_{(13)} + 2(C_{(12)} - C_{(13)} - C_{(23)})\kappa^2(u) + (C_{(22)} - 2C_{(23)})\kappa^4(u)}{(1+\kappa^2(u))^2},$$
$$\beta_u = \frac{C_{(13)} + (C_{(13)} + C_{(23)} - 2C_{(12)})\kappa^2(u) + (C_{(23)} - C_{(11)} - C_{(22)})\kappa^4(u)}{(1+\kappa^2(u))^2},$$
$$\gamma_u = \frac{C_{(11)} + 2C_{(12)}\kappa^2(u) + C_{(22)}\kappa^4(u)}{(1+\kappa^2(u))^2}, \quad \omega_u = \frac{\sqrt{\alpha_u}\sqrt{C_{(33)}}}{2(\mathrm{dn}^{-2}u-1)}, \quad \psi_u = 2\beta_u + \gamma_u.$$

The standard notations for the Legendre canonical elliptic integrals and Jacobi elliptic functions are used in the above equations. The rapidly convergent Fourier series for involved elliptic functions can be found in [8]. These series provide the most effective way of damage averaging and considerable economy in numerical work.

The limits u_1, u_2 are found from the following equations

$$\mathrm{sn}u_1 = \sqrt{\frac{2\sqrt{C_{(33)} - C_{(11)}^{-1}C_{(13)}^2}}{\sqrt{C_{(33)}^{-1}} + \sqrt{C_{(33)} - C_{(11)}^{-1}C_{(13)}^2}}}, \quad \mathrm{sn}u_2 = \sqrt{\frac{2\sqrt{C_{(33)} - C_{(22)}^{-1}C_{(23)}^2}}{\sqrt{C_{(33)}^{-1}} + \sqrt{C_{(33)} - C_{(22)}^{-1}C_{(23)}^2}}}. \tag{3.19}$$

A new variable u (the inversed amplitude) is defined by

$$\mathrm{tg}^2\varphi = \kappa^2(u) = \frac{-(4C_{(12)}C_{(33)} - 4C_{(13)}C_{(23)} - C_{(12)}C_{(33)}^{-1}\mathrm{sd}^4u) \pm 4\sqrt{\Delta}}{-4C_{(23)}^2 + 4C_{(22)}C_{(33)} - C_{(22)}C_{(33)}^{-1}\mathrm{sd}^4u}, \tag{3.20}$$

where discriminant Δ is

$$16\Delta = (C_{(33)}^{-2}\mathrm{sd}^8u - 8\mathrm{sd}^4u + 16C_{(33)}^2)(C_{(12)}^2 - C_{(11)}C_{(22)})+$$
$$+(4C_{(33)}^{-1}\mathrm{sd}^4u - 16C_{(33)})(2C_{(12)}C_{(13)}C_{(23)} - C_{(23)}^2C_{(11)} - C_{(13)}^2C_{(22)}).$$

Actually we should take care of variable u while substituting it instead of φ by verifying monotonicity for $\varphi = \varphi(u)$ (see equation (3.20)).

Complete investigation of averaged directional damage represented by the fourth rank damage tensor requires five-parametric analysis of an integral similar to I in equation (3.12). We give below the results, which are due mainly to the availability of high speed computers:

$$<I>= \int_0^1 ... \int_0^1 I(p_1, ..., p_5)\, dp_1 ... dp_5 = 1.35693417... \ , \quad \left\langle \frac{<\varsigma>}{\sqrt[4]{C}} \right\rangle = \frac{2}{\pi} <I>= 0.86385112... \ .$$

One then can compare these numerical results to those obtained by using the second rank damage tensor (see equations (3.14) and (3.15)).

By analysing the inversed amplitude ranging a primary classification of damage anisotropy in solids, based on representation (3.18) of averaged directional damage, arises. Omitting details here we discriminate only the most important type of damage anisotropy.

In view of (3.19), the length of segment $[u_1, u_2]$ (the spectral band of inversed amplitude) can be treated as a natural scalar measure of damage induced anisotropy. If the right hand sides of equations (3.19) are only slightly different, then the lower boundary inversed amplitude u_1 almost equals to the upper one u_2 (the narrow spectral band of inversed amplitude). In such a case the integral in (3.18) can be easily evaluated providing a simple computation formula for $<\varsigma>$. Confluent inversed amplitude damage spectrum corresponds to the exact equality $u_1 = u_2$.

4. Analysis of damaging processes in solids by the canonical technique

The canonical technique can be applied to investigation of damage phenomena in solids. We shall demonstrate the usability of the canonical formalism by a study of different damaging microprocesses.

4.1. Isothermal brittle damaging

As the irreversible strain energy release rate compared with that of brittle damaging is small the first term in the right-hand side of the entropy balance equation

$$\rho_R \vartheta_\varsigma \dot{s}_D = \mathrm{tr}\left(\mathbf{S}_\varsigma \dot{\mathbf{Y}}^a\right) - \sum_\beta \rho_R \left(\frac{\partial \psi_D}{\partial \varsigma_\beta}\right)_{\vartheta_\varsigma,\, \mathbf{Y}^e} \dot{\varsigma}_\beta \tag{4.1}$$

can be omitted.

In view of asymptotic formula (2.5) the specific entropy increase rate is given by the equation

$$\sqrt{\vartheta_\varsigma}\, \dot{s}_D = -\frac{G(\mathbf{Y}^e)}{\|\varsigma\|^2} \|\varsigma\|^{\cdot} + \frac{1}{\|\varsigma\|} \mathrm{tr}\left(\frac{\partial G}{\partial \mathbf{Y}^e} \dot{\mathbf{Y}}^e\right). \tag{4.2}$$

The canonical asymptotics of the specific free energy (see equation (2.5)), equations (4.1), (4.2) lead to the equation

$$\frac{\|\varsigma\|^{\cdot}}{\|\varsigma\|} + \frac{1}{G} \mathrm{tr}\left(\frac{\partial G}{\partial \mathbf{Y}^e} \dot{\mathbf{Y}}^e\right) = 0. \tag{4.3}$$

The latter equation is integrated to result in the balance equation

$$\|\varsigma\|\; G(\mathbf{Y}^e) = \text{Const.} \tag{4.4}$$

The obtained equation shows the relation between the decrease of the elastic strain and increase of damage in the course of brittle damaging.

4.2. Effect of irreversible strains on damaging

By using the same technique as in the previous case, the following equation

$$G(\mathbf{Y}^e)\,\|\varsigma\|^{\cdot} + \|\varsigma\|\,\mathrm{tr}\left(\frac{\partial G}{\partial \mathbf{Y}^e}\dot{\mathbf{Y}}^e\right) = \frac{1}{\rho_R\sqrt{\vartheta_\varsigma}}\mathrm{tr}\left(\|\varsigma\|^2\,\mathbf{S}_\varsigma\dot{\mathbf{Y}}^a\right) \tag{4.5}$$

is obtained instead of equation (4.3).

The latter equation can be integrated along the isothermal damaging process, if $\|\varsigma\|^2\,\mathbf{S}_\varsigma = \bar{\mathbf{S}}_\varsigma$, where $\dot{\bar{\mathbf{S}}}_\varsigma = \mathbf{0}$. Thus, we obtain

$$G(\mathbf{Y}^e)\,\|\varsigma\| = \frac{1}{\rho_R\sqrt{\vartheta_\varsigma}}\mathrm{tr}\left(\bar{\mathbf{S}}_\varsigma\mathbf{Y}^a\right) + \text{const}, \quad \mathrm{tr}\left(\bar{\mathbf{S}}_\varsigma\mathbf{Y}^a\right)^{\cdot} \geqslant 0. \tag{4.6}$$

Thus, the irreversible strains can intervene in the dominant brittle damaging microprocess by delaying the damage growth.

4.3. Ductile damaging

The main factor affecting the ductile damaging is the microstresses caused by inhomogeneity of the plastic flow at the microscale. Thus, the state variable $\mathbf{Y}^e$ should be replaced by the microstrain tensor $\mathbf{Y}^{\mu P}$, where μ in the superscript refers to the microdistribution. The pair of the work-conjugated microstress tensor $\mathbf{S}^\mu_\varsigma$ and the microstrain tensor $\mathbf{Y}^{\mu P}$ can be then replaced by the stress tensor $\mathbf{\Sigma}_\varsigma$ and the plastic strain tensor $\mathbf{Y}^P$ according to the microstrain energy equivalence equation

$$\mathrm{tr}\left(\mathbf{S}^\mu_\varsigma\dot{\mathbf{Y}}^{\mu P}\right) = \mathrm{tr}\left(\mathbf{\Sigma}_\varsigma\dot{\mathbf{Y}}^P\right).$$

The function $G(\mathbf{Y}^e)$ should be replaced by a monotonously increasing (as the plastic strains are increasing) up to the saturation limit G_∞ function of $\mathbf{Y}^P$. Equation (4.5) is replaced by the equation

$$G(\mathbf{Y}^P)\,\|\varsigma\|^{\cdot} + \|\varsigma\|\,\mathrm{tr}\left(\frac{\partial G}{\partial \mathbf{Y}^P}\dot{\mathbf{Y}}^P\right) = \frac{1}{\rho_R\sqrt{\vartheta_\varsigma}}\mathrm{tr}\left(\|\varsigma\|^2\,\mathbf{\Sigma}_\varsigma\dot{\mathbf{Y}}^P\right), \tag{4.7}$$

which again, if $\|\varsigma\|^2\,\mathbf{\Sigma}_\varsigma = \bar{\mathbf{\Sigma}}_\varsigma$, where $(\bar{\mathbf{\Sigma}}_\varsigma)^{\cdot} = 0$, is integrated along the isothermal ductile damaging process:

$$G(\mathbf{Y}^P)\,\|\varsigma\| = \frac{1}{\rho_R\sqrt{\vartheta_\varsigma}}\mathrm{tr}\left(\bar{\mathbf{\Sigma}}_\varsigma\mathbf{Y}^P\right) + \text{const}.$$

A universal integral can be obtained in the case of proportional loading, which often occurs in practice. By using the constitutive equations

$$Y^P_{(\alpha)} = \omega(J_\Sigma)\Sigma_{\varsigma(\alpha)} \quad (2J_\Sigma = \Sigma_{\varsigma(\nu)}\Sigma_{\varsigma(\nu)}),$$

where $Y^P_{(\alpha)}$ are the principal plastic strains and $\Sigma_{\varsigma(\alpha)}$ are the principal stresses, which are valid for a proportional loading, equation (4.7) becomes

$$\frac{d}{dJ_\Sigma}\Big[G(\mathbf{Y}^P)\,\|\boldsymbol{\varsigma}\|\Big] = \frac{1}{\rho_R\sqrt{\vartheta_\varsigma}}\,\|\boldsymbol{\varsigma}\|^2\left(\omega + 2J_\Sigma\frac{d\omega}{dJ_\Sigma}\right). \tag{4.8}$$

Assuming microstress saturation in the yielded zone and replacing in the latter equation $G(\mathbf{Y}^P)$ by the saturation limit G_∞, the following equation is obtained

$$\text{const} - \frac{1}{\|\boldsymbol{\varsigma}\|} = \frac{1}{\rho_R G_\infty\sqrt{\vartheta_\varsigma}}\int^{J_\Sigma}\left(\omega + 2J_\Sigma\frac{d\omega}{dJ_\Sigma}\right)dJ_\Sigma. \tag{4.9}$$

We conclude this section by noting that in view of the latter equation the microstress intensity decreases as the damage grows, thus the microstress free energy stored within the localized yielded zone is primarily consumed by damaging.

5. Conclusions

1) Obtained results have provided the canonical damage state variables definitions and the convenient formalism necessary for them to be usable in fracture mechanics.
2) A variant of the canonical realization of the damage description has been proposed. The variant is based on the canonical directional damage variable, derived from the canonical set of damage state variables, as opposed to that derived from raw stereological data.
3) The canonical thermodynamic damage state potentials have been obtained (in particular the Helmholtz free energy).
4) The work-conjugated stress tensor of the damage state representing the magnifying stress effect has been derived.
5) Directional damage averaging the damage represented by the second and the fourth rank damage tensors has been discussed. By the obtained integral representation of the average damage the narrow band and the confluent inversed amplitude spectra of damage have been naturally discriminated. The length of the inversed amplitude spectral band has been introduced as a natural scalar measure of damage anisotropy. In the case of narrow spectral band the simple technique of computation of the averaged anisotropic damage has been proposed.
6) The brittle and ductile damaging microprocesses have been analyzed by the canonical technique. Universal balance equations valid along various damaging processes have been obtained demonstrating a superiority of the canonical technique in the damage modelling.

References

[1] Maugin G.A. Thermomechanics of plasticity and fracture. Cambridge: Cambridge University Press, 1992. 350 pp.

[2] Maugin G.A. Internal variables and dissipative structures // J. Non-Equilib. Thermodyn. 1990. V. 15. P. 173-192.

[3] Maugin G.A. Thermomechanics of inhomogeneous-heterogeneous systems: application to the irreversible progress of two- and three-dimensional defects // ARI. 1997. V. 50. P. 41-56.

[4] Radayev Y.N. Thermodynamical model of anisotropic damage growth. Part I. Canonical damage state variables of continuum damage mechanics and thermodynamical functions of three-dimensional anisotropic damage state // J. Non-Equilib. Thermodyn. 1996. V. 21. P. 129-152.

[5] Radayev Y.N. Thermodynamical model of anisotropic damage growth. Part II. Canonical damage growth rate equations and theory of damage invariants // J. Non-Equilib. Thermodyn. 1996. V. 21. P. 197-222.

[6] Kanatani K. Distribution of directional data and fabric tensors // Int. J. Eng. Sci. 1984. V. 22. P. 149-164.

[7] Murakami S., Radayev Y.N. Mathematical model of three-dimensionalanisotropic damage state // Izvestiya Rossijskoi Akademii Nauk. Mech. of Solids. 1996. №4. P. 93-110 (in Russian).

[8] Gradshteyn I.S., Ryzhik I.M. Table of Integrals, Series, and Products. New York, London: Academic Press, 1965 (translation from the Russian ed. by A. Jeffrey).

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